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Future of Education and Skills 2030: Curriculum analysis

A Synthesis of Research on Learning Trajectories/Progressions in Mathematics

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This paper was written by Professor Jere Confrey from North Carolina State University. Alan Maloney, Meetal Shah and Michael Belcher also contributed to the preparation of this document. This paper presents a synthesis of research on learning progressions in mathematics.

Note: There are two forms of synthesis, aggregate and configurative. One (aggregate) amasses the literature summarizing the findings. While the other (configurative) shapes the literature in order to make specific points. This paper combines the two by analysing the contents of a comprehensive appendix of the relevant studies, while making more directed arguments in the body of the paper.

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Table of contents

Tables

Figures

1. The Dilemma of Learning Needs vs. Grade-Level Expectations

Nearly all countries provide guidance to schools on what mathematics to teach at each grade. In most countries, such documentation is referred to as the "curriculum."[1] The specification of "grade level expectations" (GLEs) in curriculum [standards] has been important, accomplishing three major goals: (1) identifying priorities in content to be taught, (2) describing a rate of learning which, if followed, will prepare students for a variety of college and career goals by the end of secondary school, and (3) ensuring that the introduction of topics across the different content strands of mathematics (typically number, measurement, algebra, statistics and probability, and geometry) are adequately coordinated.

As they target specific curricular topics for their grade levels, teachers face significant diversity in their students' student preparation in each class. Student preparation can range across multiple grade levels, below and above their GLEs. Because effective teaching must be proximal to the learner's current state of understanding according to *all* learning theories, there is implicit tension between complying with grade level expectations and meeting the needs of students with a range of preparation. The discrepancy between what one's students know and what is slated to be taught causes many teachers to experience a dilemma that has severe implications for student learning and the overall goals of the Education 2030 project: the dilemma of addressing students' learning needs vs. maintaining the grade-level expectations.

Students and teachers experience this in educational systems around the world. Graven (2016) describes in fairly stark terms an example of this dilemma from South Africa. Many students in upper elementary and middle school still rely on their fingers to solve many computation problems and lack opportunities to learn effective strategies for transitioning to more abstract thinking. Upper elementary teachers confront this genuine, serious lag in student understanding and strategies, and are simultaneously instructed by school inspectors to teach on grade level. As one fourth grade teacher from the Eastern Cape wrote,

> *We tell the subject advisor that I am actually at grade 2. CAPS [Curriculum and Assessment Policy Standards] says I must teach this [grade 4]. But my learners are not yet on that level. That means I have to go to grade 3 work. They [district subject advisors] said no; it is wrong. They know that some learners struggle or whatever, but we are wrong to go back to grade 2 or grade 3. We always argue about that, and then they will say, "it is from the top," and not them, and then what do you do?"*

After sharing this story from her research, Graven commented:

> *Zandi's...comments illustrate the way in which Department of Basic Education systems tend to focus on monitoring teacher compliance and curriculum coverage, rather than supporting teachers to enable high quality learning in their classrooms. Ironically rather than enabling teaching and learning, these systemic interventions seem to get in the way of the very quality that they are intended to produce. (Graven, 2016 p. 9-10)*

[1] In the United States, such documentation is referred to as the "curriculum standards."

This report (Graven, 2016) aptly captured the irony and pathos in the situation: aspirations colliding with realities. There is an urgent need to find a way to resolve this dilemma.

1.1. Addressing the Dilemma as an Open Design Challenge

Viewed in the context of the Education 2030 position paper (OECD, 2018), this collision embodies the need to redefine the learning expectations for all students via forward-leaning and transformational curriculum [standards], while improving the pathways for student populations to achieve those goals, despite vast variation in their educational preparation, resources, and opportunities to learn. I frame the dilemma of addressing learning needs vs. grade-level expectations as the following open design challenge:

Can we design **adaptive** ***systems*** that deliver curriculum and instruction that meet the needs of **all** the students, while at the same time ensuring **progress** at appropriate rates towards readiness for college and careers as conceptualised in the OECD Education 2030 vision and the draft Mathematics Competency Framework?

To meet this challenge requires creation of a dynamic system in which learning targets, associated learning paths, and related classroom assessment measures are all subject to continuous improvement and ongoing validation, in order to actively guide pedagogy and curriculum implementation. A fundamental underpinning of this dynamic system is the establishment of a shared and accessible knowledge base that can guide the development of such adaptive systems.

I propose that the emerging learning trajectories/learning progressions genre of research can contribute, first of all, to that shared knowledge base through empirical evidence on patterns of student thinking. These can in turn inform curriculum materials and instruction, tighten the feedback between teachers and students, improve inclusiveness, and accelerate student learning in order to close the gaps between curricular standards and current states of learning.

To achieve this goal, it is necessary to collect current research on learning trajectories/progressions, to synthesise the rich, dispersed research on learning into hypothesised learning trajectories formats for neglected content areas, and test and validate these learning trajectories/progressions in the context of practice. Such efforts would concurrently support design and implementation of a ***systemic approach*** to the concept of learning "progress" that both connects curricular targets to underlying LTs and provides immediate classroom access to student learning data from diagnostic assessments.

The paper is organised around seven questions:

- What is a learning trajectory/learning progression in mathematics education? (Section 2)
- Around what topics has the research been concentrated? (Section 3)
- What is known about the use and outcomes of LT/LPs in curriculum, instruction, and formative assessment? (Section 4)
- How are LT/LPs measured? (Section 5)
- What evidence is there from taking LT/LPs to scale? (Section 6)
- What is known about LT/LPs' impact on educational policy? (Section 7)
- What are the possible future roles of LT/LPs in the OECD's 2030 Vision and Competency Framework? (Section 8)

2. What is a Learning Trajectory/Progression (LT/LP) in Mathematics Education?

The concept of a learning trajectory has a long history in developmental psychology, beginning with the acknowledgement that children are *not* miniature, incomplete adults; instead, they continuously build their understanding of the world through their experiences and interactions with others, and their views of ideas evolve from naive to more sophisticated. This recognition led many scholars to an insatiable curiosity to understand how children in particular--and in fact to a degree, any naive learners--view phenomena and ideas. Piaget and his colleagues produced an entire program of research to document the ideas of children and showed the remarkable ingenuity of children in building up their understandings, which may differ markedly from an adult's more sophisticated viewpoint. Understanding this, and knowing how to bring it forth in instruction, is of critical importance for teachers, especially those who take seriously the view that "one must start where the student is."

Working from a constructivist perspective, Simon (1995) addressed the specific question of how a teacher might envision a means to help students get from their early notions to more sophisticated thinking about a target concept. In doing so, he proposed "hypothetical learning trajectories" (HLT) which included "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p.133). From this basis, the field launched a significant research effort to synthesize research on students' learning over time into learning trajectories as models of the evolution of learners' thinking along their gradual approach to targeted key ideas.

One definition of a learning trajectory, useful as a starting point, is "descriptions of successfully more sophisticated ways of reasoning within a content domain based on research syntheses and conceptual analyses" (Smith, Wiser, Anderson, & Krajcik, 2006, p. 1).

A more elaborated definition is

> *...a description of qualitative change in a student's level of sophistication for a key concept, process, strategy, practice, or habit of mind. Change in student standing on such a progression may be due to a variety of factors, including maturation and instruction. Each progression is presumed to be modal—i.e., to hold for most, but not all, students. Finally, it is provisional, subject to empirical verification and theoretical challenge… (Deane, Sabatini & O'Reilly, 2012, para. 1, cited in Graf & van Rijn, 2016, p. 166)*

The second definition recognises the importance not only of the delineation of levels, but also of the reasons for changes in levels and of recognising that individual students' paths may differ.

2.1. A Distinction in Language

Frequently, researchers and practitioners use the terms *learning trajectories* and *learning progressions* interchangeably. In mathematics education, most people use the term “learning trajectories” as derived from Simon (1995). In contrast, the term “learning progressions” dominates in science education (NRC, 2007; Corcoran et al., 2009; NGSS Lead States, 2013; Alonzo et al., 2012, Furtak & Heredia, 2014). The two terms share many characteristics (attending to student thinking, targeting big ideas, articulating a sequence of tasks). Typically when used by science educators, the term *progression* outlines longer-term curricular landmarks within curricular sequences, over years (or “long-term development of core ideas in scientific disciplines," Lehrer, 2013, p. 173), whereas in mathematics education, the term *trajectory* tends to refer to finer cognitive distinctions. Battista’s (2011) distinction, that “trajectories include descriptions of instruction, progressions do not” may likewise point to a difference between a larger-grain-size curricular framework established by progressions and more detailed cognitive processes that drive instruction in trajectories, also articulated by others (Ellis, Weber, & Lockwood, 2014).

However, the distinction between usage of the terms in mathematics and science has not been hard and fast, as evidenced by the fact that the American writers of the Common Core State Standards for Mathematics chose to use the term *progressions* as they drew on the literature on student thinking (Common Core State Standards Initiative, 2010; Daro, Mosher, & Corcoran, 2011; Confrey & Maloney, 2014; McCalllum, 2011). In another variation, Clements and Sarama (2004) chose to describe a learning *trajectory* as a combination of a developmental *progression* and an instructional sequence. Of the developmental progression, they wrote,

> *That is, researchers build a cognitive model of students’ learning that is sufficiently explicit to describe the processes involved in the construction of goal mathematics across several qualitatively distinct structural levels of increasing sophistication, complexity, abstractions, power and generality. This constructivist aspect distinguishes the learning trajectory approach from previous instructional design models that, for example, used reductionist techniques to break a goal competence into subskills, based on an adult’s perspective. (p. 84)*

For them, the instructional sequence is “composed of key tasks designed to promote learning at a particular conceptual level or benchmark in the development progression.” (p.84)

It is difficult, and, arguably unwise, to recommend the use of one term or the other given the varied reasons cited by different scholars. For this reason, in this paper, I have chosen to use learning trajectory/learning progression with the abbreviation LT/LP.

2.2. Connections to Theory and Method

The development of learning trajectories was positioned within constructivist, socio-constructivist, and/or socio-cultural views of learning. The details of those theories exert a profound influence on the meaning of learning trajectories, because they circumscribe what is likely to be the substance of the levels and the explanations of the movement between levels (Simon & Tzur, 2004; Simon, Saldanha, McClintock, et al., 2010). Learning trajectories are also tied to socio-cultural theories as culture tools that draw on the experience and background of children and that require further adaptation in contexts (Lehrer & Schauble, 2015; Penuel & Shepard, 2016; Shepard, Penuel, & Pellegrino, 2018b).

For example, major contributions to LT/LPs derive from the scholarly tradition of Realistic Mathematics Education (RME), founded by Freudenthal (Treffers, 1987; De Lange, 1987; Streefland, 1991; Gravemeijer, 1994; van den Heuvel-Panhuizen, 1996). Starting from Freudenthal's fundamental assertion that "mathematical structures are not a fixed datum, but that they emerge from reality and expand continuously in individual and collective learning processes" (cited by van den Heuvel-Panhuizen, 2003, p. 10, from Freudenthal, 1987), the RME community has leveraged a key process of "re-invention" by providing scenarios that open the possibility of inventing a mathematical idea and then moving to higher levels through "progressive mathematisation." A critical element of that theory is the "level principle," which recognises that early levels of understanding are often contextually connected and that the activity of "mathematizing on a lower level could become the subject of student inquiry on a higher level" (cited by van den Heuvel-Panhuizen, p. 13, from Freudenthal, 1987). This process of "levelling up" is a critical element of LT/LPs. Further, as in Realistic Mathematics, children should also be able to revert to a lower level. This process of moving up and down levels involves the movement by children from "models of" to "models for" (Streefland, 1991, and as cited by van den Heuvel-Panhuizen, 2003). Such detailed explanations, and others, by investigators such as Brousseau (2002), Bauersfeld (2012), and Vygotsky (1986), of how children learn increasingly abstract and generalised knowledge, are essential to the understanding and proper use of LT/LPs, as they are the propellants that connect the levels and, without which, there would be no movement.

The development of LT/LPs is typically realised through the conduct of design studies (Brown, 1992; Prediger, Gravemeijer, & Confrey, 2015) or teaching experiments (Steffe & Thompson, 2000; Confrey & Lachance, 2000) with their extended interactions in classroom settings, as opposed to laboratories, and with conditions designed to stimulate the invention and articulation of student ideas, permitting researchers to pursue possibilities for further insight into student thinking. Cobb, Confrey, diSessa, Lehrer and Schauble (2003) specified five conditions associated with design study: 1) examine explicitly a set of theoretical issues, 2) be interventionist, 3) place those theories in harm's way as a means of supporting a particular kind of learning, 4) be iterative, and 5) hold decisions accountable to design.

The resulting LT/LPs may be packaged as a set of descriptions of levels and associated tasks and/or curricular materials, but Lehrer and Schauble (2015), echoing a theme of RME, emphasised that LT/LPs are not context-free accounts of learning. Grounded in classroom studies, LTs refer more broadly to "the educational experiences that support the developing learner, and these educational experiences are delineated in the LP model both as principles that guide educational design and as mechanisms that account for learning" (p. 433). The descriptions of the tasks, as woven together to form an educational experience, are consistent with the work of Gravemeijer who refers to these domain-specific, extended, and carefully designed teaching episodes as yielding "local instructional theories." Gravemeijer, Bowers, and Stephan (2003), for example, described these local instructional theories as follows:

> *Two implications of this assumption are that we do not assume that any given sequence will play out the same in any classroom, and we do not view a proposed trajectory as a series of conceptual stages along which each individual student in the class will progress. Instead, when developing a learning trajectory, we attempt to outline conjectures about the collective development of the mathematical community by focusing on the practices that might emerge at the beginning of the sequence, then creating tools and activities that might support the emergence of other practices that would be based on increasingly sophisticated ways of acting and justifying mathematical explanations. (p 55)*

The LT/LPs are not simply psychological descriptions of learning but are, rather, situated in a larger conceptualisation of the roles of students and teachers in overall classroom learning ecologies. Likewise, the tasks associated with levels are not simply stimuli for responses, but involve setting up conditions for instructional participation and learners' activity. Thus, the tasks are situative in that, while they may introduce disciplinary content to learners, they must also help relate those contexts to their experience and background (Bang & Medin, 2010; Shepard, Penuel & Pellegrino, 2018a).

Ensuring sufficient attention to possible cultural connections of LT/LPs remains a challenge for many LT/LP scholars, according to Delgado and Morton (2012). They warned that in order to promote equity in relation to LT/LP scholarship, researchers need to pay more careful attention to conducting research in diverse settings, including all student ideas, understanding what students from diverse backgrounds bring to instruction, and providing sufficient supports in LT/LP investigations for learners at the lower levels.

An overall expression of the goal of the LT/LP is to:

> *provide a horizon of development, a vision that can guide instruction over the long term. Learning is emergent and therefore, will always be variable, but having some notion of what to anticipate is critical for managing complexity constructively. (Lehrer and Schauble, 2015, p. 435)*

This characterisation of LT/LPs is therefore that of researchers' theory-based and empirically-driven models of the progression, over time, of students' diverse and increasingly sophisticated thinking, grounded in and emerging from rich classroom ecologies. Significant amounts of time and resources are required to create these "design-based" learning trajectories/progressions. This raises a methodological challenge for the purposes of building a scalable adaptive dynamic system: whether a more parsimonious means of LT/LP development is feasible.

2.3. LT/LPs Are Not Stage Theories

Psychologists also address issues of sequence in learning, the best known of which is the Piagetian stage theories. A stage theory, strictly hierarchical, requires that children master one level before proceeding to the next. However, most LT theorists (Lehrer & Schauble, 2015; Maloney, Confrey & Nguyen, 2014; Battista, 2011) reject a strict stage theory as a model for LT/LPs. LT/LP theorists recognise that the levels are not rigidly sequenced (NRC, 2007, p. 221), and that students advance and fall back, making steady progress when viewed over time (Middleton, Flores, Carlson, Baek, & Atkinson, 2003; Stephens et al., 2017). These theorists regard the ordered sequence of levels as “expected probabilities” (Confrey, Maloney, & Nguyen, 2014, p. xvii) and “benchmarks of complex growth that represent distinct ways of thinking” (Clements and Sarama, 2014, p. 14).

LT/LPs can be likened to ladders, with students ascending a rung at a time, but this image can inadvertently reinforce the stage theory model. Battista (2011) refers to the levels as a variety of plateaus of different heights. To capture the flexibility in students’ paths, Confrey & Toutkoushian (in press) use the metaphor of a climbing wall, which represents a conceptual space that presents both handholds and obstacles, and supports multiple starting points and routes.

2.4. Epistemological Objects in the Levels

The levels of a LT/LP are comprised of certain kinds of “epistemological objects” that make sense from the learner’s point of view and help in the process of coming to know. The characters of these objects (and the tasks related to them) distinguish learning trajectories from a solely logical deconstruction of a mathematical idea. Objects at different levels may represent shifts in learners’ attention, reflect student inventions, signal the rejection, extension, or transformation of prior levels, draw connections to everyday experience, and may depend on the introduction of new language, needs for justification, or consideration of a larger range of cases.

The levels may contain naive and partial conceptions. For example, when students collect data on, for instance, the arm spans of their classmates and display them on a number line, the students’ displays often include an ordered set of values, with repeated values displayed in stacks (a case-values plot). Those displays, however, typically ignore gaps in the data values (Lehrer, Giles, & Schauble, 2002). From the student’s point of view, the case-value plot accomplishes their goal of presenting and ordering all the data; from an expert’s standpoint, the child has not yet fully distinguished scale from data points.

Levels include invented or limited representations utilising particular features (i.e. tables of data). For example, van den Heuvel-Panhuizen (2003) noted that the Mathematics in Context learning-teaching trajectory on percent described the evolution of the percent bar as beginning “with a qualitative way of working, with percentages as descriptors of so-many-out-of-so-many situations,” (p. 18). Tasked to represent an auditorium’s “fullness," the percent bar emerges from learner’s sketched rectangular depiction of full and empty rows. It is followed by the gradual development of an “occupation meter” that evolves into a fully-featured percent bar and eventually a double number line. As the representations matured, the mathematical concepts became increasingly sophisticated beginning with simple informal percents, to benchmark percents, to operationalising 1%, to calculating *x*% of, and then to percent increase and decrease.

Student-generated strategies for solving a task frequently populate LT/LP levels. For example, Outhred and Mitchelmore (2000) described how third grade students build an understanding of area as the product of length and width beginning with early strategies of haphazardly filling in the space with overlapping squares and gaps. Later this strategy is replaced with complete and ordered rows and columns. As learners re-conceptualise the task to be one of finding the total number of square units, they may only need to place the squares along length and width to predict the total or they may use tick marks for the same purpose. Strategies are indicated by their variations in efficiency as students reconstruct the value and utility of more complex approaches.

Novel ideas, not in evidence in mathematical texts, emerge. Confrey (1995) reported on a fourth-grader's invention of "the littlest recipe" to describe the smallest whole-number ratio equivalent as a term which was later renamed "base ratio." While mathematically equivalent to rewriting the ratio as a fraction in simplest form, the expression of it as the "littlest recipe" facilitated its use in building up to equivalent ratios in tables and for transitioning to slope in graphs.

The diversity of kinds of epistemological objects that comprise learning trajectories make it clear that levels differ qualitatively. The emergence or epistemological objects requires settings in which students are given challenging but reachable tasks and are encouraged to propose ideas, invent representations, try strategies, and express ideas. They represent opportunities for researchers to attend to student "voice" and, in doing so, to rethink their own expert "perspective," thus creating a voice-perspective dialectic (Confrey, 1998). Or as Lehrer and Schauble (2015) wrote, "LPs are a way to restructure and rethink the content and/or the sequence of the subject matter that is taught. They often serve as proposals to shift our view of what it means to understand an idea..." (p. 433). Table 1 summarises qualities of learning trajectories and distinguishes those from common mis-perceptions about learning trajectories.

Table 1. Qualities of learning trajectories (left) and mis-perceptions (right)

What Learning Trajectories *Are*	**What Learning Trajectories Are *Not***
Domain-specific models	General or universal principles
Expected probabilities	Stage theories
Empirically-based models of student thinking	Logico-mathematical deconstructions
Based in students' thinking	Based in opinions of experts in mathematics
Elicited by rich or novel tasks	Derived from typical exercises
Include strategies, reasons, explanations and cases	Sub-goals of the target
Include exploring misconceptions	A means to avoid errors
Ordered by increasing sophistication	Ordered by difficulty
Connected to big ideas over the long term	Curriculum material
Evolving	Fixed

2.5. LT/LPs and Mathematical Practices

The majority of LT/LPs target a concept or big idea, but it is also important to have LT/LPs for mathematical processes or practices (Empson, 2011). For instance, Stephens and Armanto (2010) describe a learning trajectory for relational thinking in primary school. They defined "relational thinking" as occurring when students "consider a number sentence as a whole, then analyse and find the structure and important elements or relationship to generate productive solutions" (Molina, Castro, & Mason 2007). The study consisted of an analysis of the learning trajectory within the Japanese Shoseki curriculum (Hironaka & Sugiyama, 2006).

2.6. Grain Size

Researchers differ in the grain size of the levels. Those grain size differences typically reflect differences in the purpose for the LT/LP. Bernbaum Wilmot, Schoenfeld, Wilson, Champney, & Zahner (2011) built a learning progression for functions consisting of six levels--from 0) pre-algebraic to 1) prestructural, 2) unistructural, 3) multistructural, 4) relational, and 5) extended abstraction--to describe conceptual growth from 6th to 12th grade. In contrast, Confrey, Maloney, Nguyen, and Rupp (2014) built one on equipartitioning with fourteen levels, covering three primary grades. The functions progression was developed to ascertain a student's progressive degree of college readiness, while the purpose of the equipartitioning trajectory was to assist teachers in detecting differences in student thinking as a means to strengthen their learning in daily instruction. The choice of grain size is influenced by how the LT/LP is used, for instance, for extended curricular development, to evaluate readiness, to diagnose student progress or readiness, or inform everyday instruction.

2.7. Five Commitments Shared by LT/LP Theorists

In summary, the concept of a LT/LP has emerged as a means to capture and communicate the theoretically-driven empirically-based patterns of learning as students move from naive to sophisticated thinking. They differ fundamentally from solely logical deconstructions of the mathematical ideas, although conducting such analyses can serve as one resource in conjunction with rich understandings of students' experiences and cultural resources. Learning trajectories depend on learning theories that value the investigation of a student's ideas and require instructional settings and tasks devoted to stimulating and exploring a learner's ideas. All LT/LP theorists working from the HLT foundation share five commitments: 1) LT/LPs are conjectures that involve modelling learning processes with respect to specific domains within a constructivist environment and a socio/cultural perspective; 2) articulation of the levels must be linked to theoretical mechanisms that account for transitions to more sophisticated levels; 3) instruction plays an essential role in students' progress along LT/LPs; 4) "all students will follow not one general sequence, but multiple (often interacting) sequences" (NRC, 2007, p. 221); and, 5) the LT/LPs will become apparent to the degree to which the instructional interactions support the emergence of student thinking. LT/LPs are investigated over extended periods using design studies and teaching experiments.

3. Around What Topics has the Research been Concentrated?

Papers were collected in order to summarise the distribution of topics of LT/LPs among current research. During the initial review, papers were included if they used the phrase “learning trajectory,” “learning progression,” “instructional theory,” “conceptual change,” “developmental progressions,” or “longitudinal analysis,” and were published within the last 15 years. Reviews of the research on LT/LPs provided a rich source of relevant literature (Lobato & Walters, 2017; Groff, 2017). The resulting 124 papers were categorised as: 1) presenting an LT/LP for a content domain, 2) general theoretical papers, or 3) applications of LT/LPs to curriculum, teaching, or informal assessment. Of the 124 papers, 85 discussed specific LT/LPs. Some of the 85 papers reported on the same LT/LP, often written by the same author(s). In order to examine the distribution of different LT/LPs, duplicates were eliminated, reducing the total number of distinct LT/LPs to 75. The distinct LT/LPs are listed in Appendix A, and the related set of references is listed in Appendix B.

All of the remaining analyses were conducted on this set of 75 LT/LPs. Table 2 describes the distribution of LT/LPs by grade level. Equal percentages of LT/LPs were located in elementary and middle grades with a very small number focused on high school topics.

Table 2. Summary of LT/LPs (total = 75) by grade level.

Pre-K	Pre-K and elementary	Elementary	Elementary and Middle	Middle only	Middle and High School	High School only
0	0	30 (40%)	8 (11%)	30 (40%)	4 (5%)	3 (4%)

A second analysis (Table 3) revealed that a significant majority of the LT/LPs published in the literature do not use formal psychometric models, although a few researchers have conducted multiple studies using measurement models with their LT/LPs; given the coding scheme by LT/LP, those are counted once. Further, the number of published papers on these topics seems to lag behind the current activity involving psychometric models.

Table 3. Prevalence of formal psychometric models in mathematics LT/LP database

Formal measurement	No formal measurement
14 (19%)	61 (81%)

The final analysis examined the topics in which LT/LPs were concentrated (Table 4). It revealed that the majority of LT/LPs published are on number concepts (45%), followed by algebra, including early algebra (24%), then measurement (12%) and then probability and statistics (11%). Geometry has the fewest LT/LPs (8%), but the van Hiele-based research is not included in the analysis (though it could be classified as a domain-specific progression).

Table 4. Distribution of LT/LPs (total = 75) by topic.

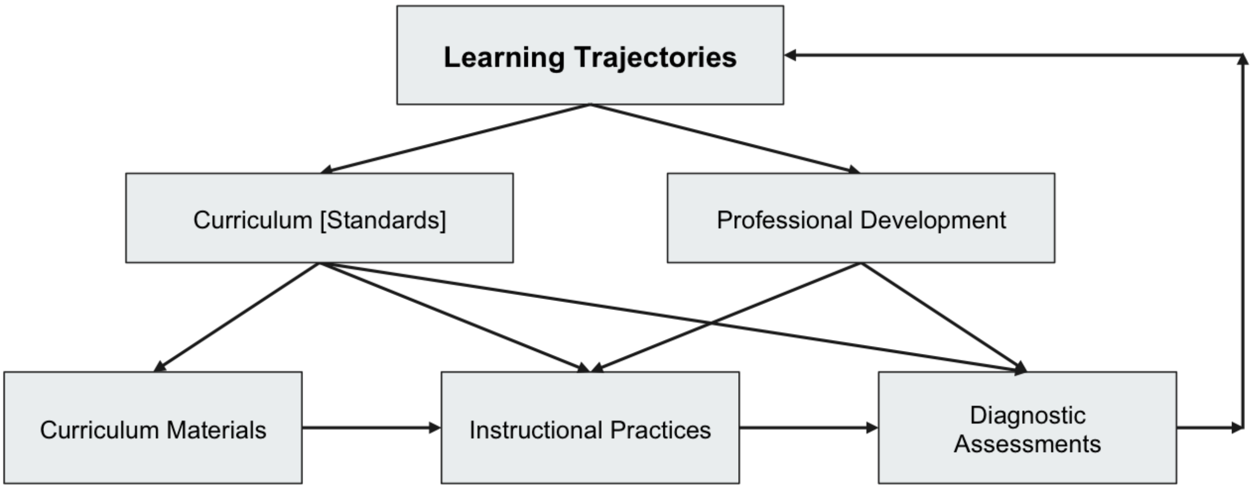
Number	Measurement	Geometry	Algebra and Functions	Probability and Statistics
34 (45%)	9 (12%)	6 (8%)	18 (24%)	8 (11%)

This analysis confirms reports by others (Groff, 2017; Daro et al., 2011) that the distribution of topics covered by LT/LPs is spotty. Should a major effort be proposed to build a next generation learning approach resting on a foundation of LT/LPs, a major research effort would be required.

4. What is known about the Use and Outcomes of LT/LPs in Curriculum, Instruction, and Formative Assessment?

LT/LPs are models of student thinking that sit within a larger theoretical and methodological paradigm of constructivism and socio-cultural perspective. These frames are essential to provide scaffolding for the design and conduct of the design studies and teaching experiments, and to generate the data and the interpretive frameworks to develop the results. LT/LPs do not affect student learning directly, rather they are mediated through the variety of educational components shown in Figure 1. These practices are viewed as places where LT/LPs are applied.

Figure 1. A rudimentary logic model for the use and impact of learning trajectories on educational components.



4.1. LT/LPs and Curriculum Materials

LT/LPs can be used to conduct content analyses of curriculum materials (NRC, 2004). The purpose of many content analyses (Adams Tung, Warfield, et al., 2000; AAAS, 1999a, b) is to examine the alignment of curricular materials with curriculum [standards]; however, other lenses, including learning trajectories, can be used. So far, these analyses are typically conducted on the materials only (Nguyen & Confrey, 2014; Olson, 2014; Stephens and Armanto, 2010).

A second use of research relating LT/LPs and curriculum investigates the question “to what degree are curricular goals met at each grade level based on the students’ progress along a LT/LP?” The question is investigated by postulating relationships between curricular targets and levels of a LT/LP, and determining their correspondence using a design based on cross-sectional analysis. For instance, Ayalon, Watson, and Lerman (2016) analysed the Israeli functions curriculum by grade (grades 6-12), identifying key ideas at each grade. Using a learning progression for functions derived from the research on functions by

Leinhardt, Zaslavsky, and Stein (1990), the authors generated a sequence of three open-ended tasks designed to measure student understanding of rate of change, covariation, and correspondence in various situations. Responses to the tasks were collected cross-sectionally by grade level (7^{th}-11^{th}) by students categorised as either high- or average-achieving learners. Results indicated high-achievers lagged a grade below expectations and average-achievers lagged from 1-4 grades below. Though they concluded that rate of change and correspondence of functions were eventually adequately learned by all students, this was not true for relating functions to situations. Investigating the degree to which students meet grade level expectations in relation to their progress on LT/LPs provides a reasonable means to monitor the extent of the gap between students' learning needs and the grade-level expectations.

4.2. LT/LPs, Instruction, and Professional Development

Often the most direct application of the research on LT/LPs is the study of their use in classroom instruction. Teachers can use LT/LPs to help plan instruction, guide the selection of classroom tasks, and facilitate classroom discussions. The LT/LPs can also help teachers interpret evidence and make informed judgments about how students in a class might differ in terms of their understanding of key concepts and practices. Instructional leaders, coaches, and administrators can use the LT/LPs as a foundation for content-based professional development, to deepen teachers' understanding of how students learn.

Research on instructional perspectives on LT/LPs has focused on how teachers use LT/LPs to modify instruction. Numerous studies were conducted using the equipartitioning LT/LP with 56 preservice teachers (Mojica, 2010), practicing teachers (P. H. Wilson, 2009), and with 24 teachers at the same elementary school (Edgington, 2012; Wilson, Mojica, & Confrey, 2013; Wilson, Sztajn, Edgington & Confrey, 2014). From observation and analysis of the types of instructional activities affected by teacher's experience with the LT/LP, the researchers reported increases in the mathematical knowledge for teaching (MKT), especially regarding their pedagogical content knowledge (PCK) and, to a lesser but still significant degree, their subject matter knowledge for teaching (SMK) (Ball, Thames, & Phelps, 2008).

Suh and Seshaiyer (2015) conducted a teaching experiment with six teachers from a cohort of 37 3^{rd}- to 8^{th}-grade teachers who were studying an algebra LT/LP. The professional development was designed for vertical teaming and the use of lesson study. An algebra lesson was modified for use at three different grade levels. The researchers found that the teachers strengthened their content knowledge, including their understanding and use of conceptual principles, their ability to use modelling strategies, the way they selected and modified problems, and their anticipation of students' misconceptions, strategies, and representations. They also developed more assessment practices around focusing on multiple levels of the LT/LP.

Studies focusing on the relationship between different kinds of epistemological objects in LT/LPs and instruction contribute further to understanding the view of learning associated with LT/LPs. For example, Wiig, Silseth, and Estad (2018) conducted a study in a lower secondary school in Norway of intercontextuality in learning trajectories. Using interactional analysis, they studied how and to what extent teachers drew upon the relationships between everyday and scientific knowledge (including mathematics), to form dialogic interactions. They found that frequently teachers superficially draw those connections, and concluded that teachers "need to be more conscious of the challenges of framing interactions that students consider significant as resources for their engagement in the social creation of intercontextuality" (p 17). They drew on Engle's (2006) work to claim that "creating intercontextuality involves not only knowing but also doing and that doing inherently entails the exercise of human agency" (p. 18). Their study demonstrated the potential connections among LT/LP levels connected with learner's experience, learner-centred instruction, and student agency.

Another study of instruction connected epistemic content of LT/LPs and agency (Shinohara & Lehrer, 2018). Situated in a large-scale LT/LP study of sixth grade students (Lehrer, Kim, Ayers & Wilson, 2014), the study reported on observations and interviews of students as the students participated in forms of epistemic practice (Cetina, 2009), involving visualising, measuring, and modelling variability. When asked to describe how their thinking had changed over the course of instruction, students identified four aspects of the classroom activity: 1) how students' invention of representations, measures, and models of variability made variability evident to them, 2) how critique offered opportunities to decipher and perhaps contest the inventions of others, and so elaborate the grounds of knowing, 3) how recontextualisation provoked consideration of how previously developed ways and means could be productively deployed to interpret new contexts of variability, and 4) how coordination consisted of developing hybrids of practices as students learned to orchestrate these practices to interpret different sources and types of variability, such as sample and sampling variabilities. They further describe how these common aspects also emerged to foster individual differences in values and dispositions. The study illustrates how LT/LPs can be leveraged through instructional designs, not just to learn a disciplinary concept as the target of an LT/LP, but to support students to participate with agency within an authentic statistical practice with opportunities for self-expression and critique (Shinohara & Lehrer, 2018, p. 20).

4.3. LT/LPs and Classroom/Formative Assessment

LT/LPs are also applied within the context of "classroom assessment" (NRC, 2003; Pellegrino, Chudowsky, & Glaser, 2001; Pellegrino, DiBello, & Goldman, 2016). Classroom assessments are defined as "assessments for supporting students *while* learning by providing relevant, timely, detailed, and actionable feedback on their current progress...to guide instructional decision-making" (Confrey, Maloney, Belcher, et al, 2018). They build on the foundation of formative assessment (sometimes called "assessment for learning" (Carless, 2017; Black & Wiliam, 1998; Black, Harrison, Lee, Marshall, & Wiliam, 2004) or "assessment as learning" as a "learning process which scaffolds students' agency and ability to self-regulate their learning" (Fletcher, 2018, p. 4). Heritage (2007, 2008) identified learning progressions as one of four key elements that teachers must understand in order to effectively use formative assessment practices. The learning progressions act in concert with the other three core elements of: 1) identifying the gap between a teacher's learning goals and the knowledge state of the students, 2) using feedback, and 3) promoting student involvement. She argued for assessment practices to create a system to scaffold student learning around an appropriately-sized zone of proximal development (Vygotsky, 1986), provide relevant and timely feedback on student progress, and involve students centrally and reflectively in the process. Heritage (2008) conjectures that the fine grain-size of LT/LP levels can assist students in monitoring and evaluating their own progress in order to "have a manageable way to be self-reflective about their own learning while they are learning." (p.7.) Classroom assessment is connected to the self-regulated learning, and, increasingly, concept of agency[2], in which students are viewed as actors who “make choices and whose actions shape assessment practices in both anticipated and unexpected ways” (Adie, Willis, & van der Kleij, 2018, p. 1). Because LT/LPs can provide students and teachers access to patterns in student reasoning and uses of language, they can contribute to strengthening reciprocal relations between teachers and students (Fletcher, 2018), supporting “agentic engagement" (Reeve, 2012; Reeve & Tseng, 2011). These can contribute to the implementation of formative assessment practices such as peer- and self-assessment, increasing students’ understanding of the criteria by which they are being judged, and collecting and identifying evidence of their own learning in order to develop academic values such as competency, autonomy (self-determination), and relatedness (Adie, Willis, & van der Kleij, 2018, p.3).

The use of formative assessment practices with learning trajectories has also been accompanied by explicit use of student work artefacts to guide the teachers’ attention to variations in student’s levels of thought (Petit, 2011; Lehrer et al, 2014; Blanton, Brizuela, Gardiner, Sawrer, & Newman-Owens 2015; Siemon, Barkatsas, & Seah, 2019). In these cases, analysis of those artefacts has been shown not only to act as evidence of progress along the trajectory (Ayalon et al., 2016), but to act as a means to achieve profound amounts of professional development (Suh & Seshaiyer, 2015). Yerushalmy, Nagari-Haddif, and Olsher (2017) emphasise the value students’ examination of each other’s work as a central part of the learning process, and have built technologies to promote the exchange of those artefacts and facilitate careful sequencing by teachers.

Supovitz, Ebby, and Sirinides (2013) developed a measure of learning trajectory-oriented formative assessment called TASK that measures four domains: teachers’ knowledge of mathematics, analysis of student understanding, knowledge of mathematics learning trajectories, and instructional decision-making in addition, subtraction, fractions, proportions and algebra. They scored teacher responses on a four-point rubric ranging from general, to procedural, conceptual and learning trajectory-based responses. More than 1200 teachers in K-10 from largely urban and urban fringe districts responded to the survey. To evaluate the learning trajectory proficiency of teachers, they were asked to order a set of student responses relative to three categories, evidence of solid, transitional, or no mathematical thinking; their responses were subsequently evaluated relative to the reasoning provided for the ordering. In grades 6-8, only 14% selected the correct order, and 24% ranked one of the lowest two responses as advanced; however, in 9-10 grades, 81% correctly ordered the tasks and more than half identified the most sophisticated thinking. Few teachers, however, could articulate the reasons for their ranking developmentally. Investigating teachers’ instructional responses, the researchers found only between 2%-9% (across the different grade levels) choose a learning trajectory-based response. The data suggest that these American teachers at lower grades are excessively procedural in their orientation, and that while middle and early high school teachers are more conceptual in their orientation to examining student work, few take a learning-trajectory approach, which would allow them to view current learning in relation to a bigger picture of development.

Leahy and Wiliam (2011) offered a contrasting approach to learning progressions from within a perspective of formative assessment, describing learning progressions as simply an agreed-upon view of “what it is that gets better when someone gets better at something”

[2] Emirbayer and Mische (1998) define *agency* as: temporally constructed engagement by actors of different structural environments - the temporal-relational contexts of action - which, through the interplay of habit, imagination and judgment, both reproduces and transforms those structures in interactive response to the problems posed by changing historical situations (p. 971).

(p.1). They grounded their approach in Gagné's learning hierarchies as "a set of specified intellectual capabilities having, according to theoretical considerations, an ordered relationship to each other" (Gagné, 1968 p. 2, cited in Leahy & Wiliam, 2011). They recounted Denvir & Brown's (1986a, b) experiment in number skills, in which the performance of 41 students on 47 skills was rank-ordered to produce an S-P table (Sato, 1975), whose analysis led to an empirically-supported, seven-level tiered hierarchy. After subjecting those empirically-derived results to a logical analysis, Denvir and Brown claimed to have produced a useful and valid local progression. Leahy and Wiliam suggested that a similar protocol could be developed for local use by teachers to form their own local learning progressions (which, they suggested, would be widely and willingly implemented because the progressions would already be embedded in the day-to-day work of the teachers). The example provides a provocative contrast with many approaches to LT/LPs reported in this paper, drawing as it does from a contrasting theoretical approach (behaviourism as opposed to constructivism), and suggests one can build progressions informally through the local activities of teachers, a position consistent with a formative approach. While an ordering by difficulty would result, and levels would be formed inductively from the clusters in that ordering and likely have strong credibility with teachers, there is a substantial risk that the tasks will not differ substantially from ordinary classroom exercises. As a result, the resulting LT/LPs may pay inadequate attention to the need to elicit a broad range of student ideas (including novel or unexpected ones), as conceptualised in the design-based meaning of LT/LPs. While some would see this as a reason to disqualify these as LT/LPs, one can also recognise value in their solid empirical connections to the work of practitioners and their potential to contribute to, and potentially accelerate, the overall body on work on LT/LPs. To signal the difference in this use of LT/LP from the design-based LT/LPs, these are included and labelled as "classroom behaviour-based LT/LPs"[3].

[3] This classroom behaviour-based LT/LPs bears a resemblance to a "bottom-up" approach described by Heritage (2008) as involving "curriculum content experts and teachers in developing a progression that is based on their experience of teaching children (p. 12).

5. How are LT/LPs Measured?

A major thread of LT-oriented research in assessment involves the development and validation of assessments of learning trajectories using formal measurement approaches. These assessments, unlike high-stakes or summative assessments, will be specific to the curricular topics currently being taught, and the results returned to students and teachers in a timely way. Their primary purpose is to provide teachers with information about the progress of their classes along the LT/LPs, in order for teachers to modify their instructional strategies to help more students be successful. The foremost questions to be asked of the measurement approach, then, is "what information can the measurement of LT/LPs provide and how can that information inform decision-making in a timely and relevant way?" More precisely, the questions might be:

1. (For a teacher) What ideas do my students hold and what do these tell me about the quality of their understanding?
2. (For a teacher) What is the range and distribution of my students' ideas?
3. (For a teacher) Where are my students along a LT/ LP? (to know how to focus my efforts and provide the appropriate learning support to the right students in a timely and effective way)
4. (For a student) Where I am along an LT/LP? (so I can have some confidence in what I know, and understand what I need to work on)
5. (For both) Are my students (am I, as an individual student) learning?

One way to consider what an LT/LP assessment provides that is different from regular assessments is to contrast the information gleaned from a domain-sampling test and a LT/LP test (Briggs & Peck, 2015). A domain-sampled test simply draws items from different areas of the content, whereas an LT/LP assessment is structured around the sequence in the LT/LP. Because of the structure of a LT/LP, the information provided to the teacher can be structured and compact, showing the students' degree of correct responses on specific levels, and thereby identifying which students need help on which levels. See Figure 4 (Section 6) for an illustration. If standards of good measurement are applied, the information will be valid and fair, and to some degree, reliable.[4]

5.1. Approaches to Building Measures of LT/LPs

Wilson and colleagues have played a major role in the development of methodologies for constructing measures of learning progressions. Their Berkeley Evaluation and Assessment Research (BEAR) assessment framework is grounded in four "principles of sound measurement:" 1) a developmental perspective, 2) a match between instruction and assessment, 3) the generating quality of evidence, and 4) management by instructors to allow appropriate feedback, feed-forward, and follow-up (M. Wilson, 2009). The general application of these principles to create an LT/LP assessment involves creating a construct map consisting of a set of progress levels for a given subject, undertaking item design, delineating an outcome space, and producing a Wright map of item difficulties from a

[4] The meaning of reliability will probably have to be reconsidered from within the LT/LP perspective, in which rapid student intellectual growth is desirable.

Rasch Item Response Theory (IRT) analysis. In mathematics and statistics, the BEAR method, a modified version of it or another statistical approach, has been applied to create measures of LT/LPs in functions (Bernbaum Wilmot et al, 2011), area measure (Lai, Kobrin, DiCerbo, & Holland 2017), proportion and rational numbers (Carney & Smith, in press; Ketterlin-Geller, Shivraj, Yovanoff, & Basaraba, 2019), algebra readiness (Ketterlin-Geller, Shivraj, Basaraba, & Schielack, 2018), quadratic functions (Graf, Fife, Howell & Marquez, 2018), data modelling (Lehrer et al 2014), and geometric similarity (Shah, 2018), to mention a few.

5.2. Validation of Measures of LT/LPs

Other measurement teams have described their work as "validations of LT/LPs," using this language to include the processes of both building and validating trajectories. Graf and van Rijn (2016) suggest that the LT/LPs that result from the design-based studies and syntheses of the literature be viewed as "provisional," and require "empirical verification and theoretical challenge" (p. 167). They situate their studies in the larger context of empirical verification, consideration of rival hypotheses, the recognition that one does not validate the measure itself but rather the *use of* the measure (Messick, 1989), and the value of a larger interpretation/use argument (Kane, 2013).

The majority of measurement approaches to the validation of LT/LPs include many or all of the following steps:

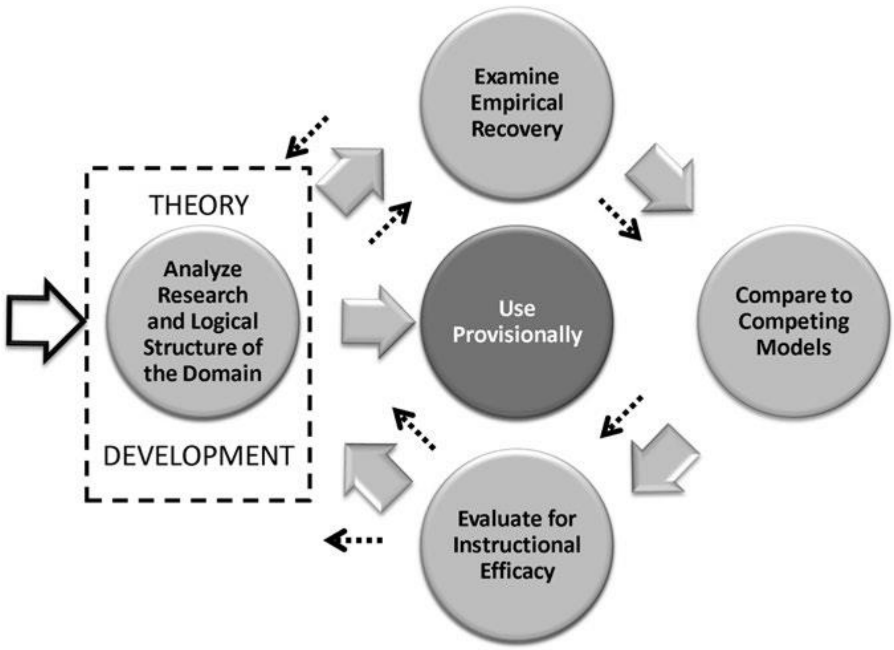
1. Synthesise existing research to describe LT/LP levels,
2. Obtain external reviews of the LT/LPs by experts,
3. Develop tasks to map to the LT/LPs,
4. Conduct think-aloud interviews of students solving the tasks to look for construct validity,
5. Conduct large-scale cross-sectional data collection for "empirical recovery" of the LT/LP, and
6. Examine consequentially whether the trajectories provide a meaningful lens into the understanding of student responses.

"Empirical recovery," the process of seeing if the data from an assessment support the hypothesised structure of an LT/LP, requires careful examination of the theoretical assumptions concerning task design, test assembly, test scoring and sample selection (Bennett, 2015). It also depends on the process of selecting a psychometric model that is influenced by how the LT/LP is conceptualised (Graf & van Rijn, 2016). For instance, most design-based LT/LP theorists agree that levels are not rigidly compartmentalised, that students may fluctuate across levels, and that particular numerical values and unfamiliar contexts in tasks may cause performance to vary (Graf & van Rijn, 2016), but they hold these beliefs to varying degrees. Depending on whether levels and their scoring are viewed as discrete or continuous (or a mixture), psychometric modelling approaches may include latent class models (Steedle & Shavelson, 2009), cognitive diagnostic modelling approaches (CDM) or IRT (Mislevy, Almond and Lukas, 2003; De Boeck, Wilson, & Acton, 2005). Briggs and Peck (2015) examined various approaches to the measurement of LT/LPs, evaluating their potential to inform and define models of growth.

Recovery also requires a careful examination of the relationship of the items to the LT/LP. The process of empirical recovery will likely result in suggestions for revisions and modifications for the LT/LPs. For example, in building and validating a diagnostic measure for student learning trajectories in middle grades mathematics, Confrey, Toutkoushian, and Shah (2019) followed the application of IRT models with an iterative application of linear regression to identify potentially non-conforming items. They postulated three sources of variation as explanations for potential non-conformance of items: construct-irrelevant variation, intra-level variation, and inter-level variation. Their team, with expertise in both the learning sciences and psychometrics, examined the data (nature of the sample, distribution of responses, performance of similar items) to decide to either remove the item, adjust its difficulty, flag and retain it as an outlier, move it to another level, resequence the level, or occasionally decide to rebuild the LT/LP. They factored in consideration of the students' opportunity to learn about the construct underlying the level. The team thus viewed validation as an ongoing process (Confrey & Toutkoushian, in press; Confrey, McGowan, Shah, et al., 2019; Shah, 2018).

Graf and van Rijn (2016) proposed a model for the validation of LT/LPs (Figure 2). They emphasised the importance of considering competing models and recognising the field's need for further elaboration on the evaluation of instructional efficacy.

Figure 2. Proposed cycle for validating a learning progression, from Graf & van Rijn (2016).



5.3. Distinguishing between a LT/LP and its Measure

A thorny definitional issue emerges in the context of measurement and the use of the terminology "learning trajectory/learning progression." Some measurement experts describe the result of their work as building a LT/LP itself, not as building a measure of a LT/LP. For example, Adams, Jackson, & Turner (2018, p.2) defined a learning progression as "a scale that defines the constructs that constitute educational progress in a particular domain (say, reading or mathematics)." This conflates the concept of a scale of educational progress with the constructs that reside in the progression. Naming such measures as LT/LPs risks leaving the field unable to distinguish between LT/LPs built through extensive, theoretically-rich design studies, on the one hand, and those which are primarily measures-based after synthesis of existing research, on the other.

The value of analogous distinctions has been established in a variety of fields. Geometers distinguish between the line segment AB and the measure of that segment (mAB), or between an angle ABC and the measure of the angle (m⦣ABC). The distinction signals the use of axioms of geometry and axioms of measurement. Scientists distinguish between the concept of "heat"[5] and its measurement scale, "temperature." The concept "heat" is used to create scientific explanations, while "temperature" is used as a measurement used in empirically testing those ideas. In science, it is essential for the concept and the measure to be used conjunctively, with the recognition that errors and opportunities for advances may come from the concept, the measure, or their interaction.

To address the distinction among LT/LPs, those built from theoretically-based, empirically-grounded design studies will be referred to here as *design-based LT/LPs*, while those built as measures will subsequently be referred to as *measures of LT/LPs*, written as mLT/LP. Some design-based LT/LP scholars question the adequacy of mLT/LPs, arguing they distort the qualitative aspects of the levels (Battista, 2011; Stacey & Steinle, 2006), but most see that finding ways to measure LT/LPs contributes to the effort to scale their use.

Distinguishing LT/LPs and mLT/LPs offers numerous advantages, the first of which is ensuring that both design-based LT/LPs and measures of LT/LPs meet their respective theoretical and methodological standards. For example, design-based LT/LP methodology may employ clinical interviews (Opper, 1977) while measure-based approaches use think-alouds (Ericsson & Simon, 1993), and the two approaches fundamentally differ as one is designed to explore student thinking while the other is designed to check if the reasoning associated with the task reflects the underlying measurement construct. This separation also permits researchers to use design-based insights to inform the development and elaboration of the measures, while simultaneously allowing the results from LT/LP assessments to shed light on the design-based LT/LPs. For example, data from assessments can show that an item or set of items is more difficult than expected, which can lead the development team to adjust levels, clarify the meaning of a level, or fine-tune particular items. Finally, recognising the difference between LT/LPs and mLT/LPs can foster mutually respective exchanges and collaborations between the learning scientists and measurement specialists and clarify the underlying LT/LP.

5.4. LT/LPs as Deep Collaborations among Learning Scientists, Practitioners and Measurement Experts

A collaboration among Lehrer, Schauble, Wilson and colleagues provides an example of a project that has combined learning sciences and measurement perspectives within a program of research around data modelling (Lehrer et al., 2014; Lehrer, 2013). Lehrer articulated a model of data modelling and conducted years of classroom-based design studies around the constructs of statistical reasoning, encompassing the development of young learners' understanding of variability, distribution, measures of centre and

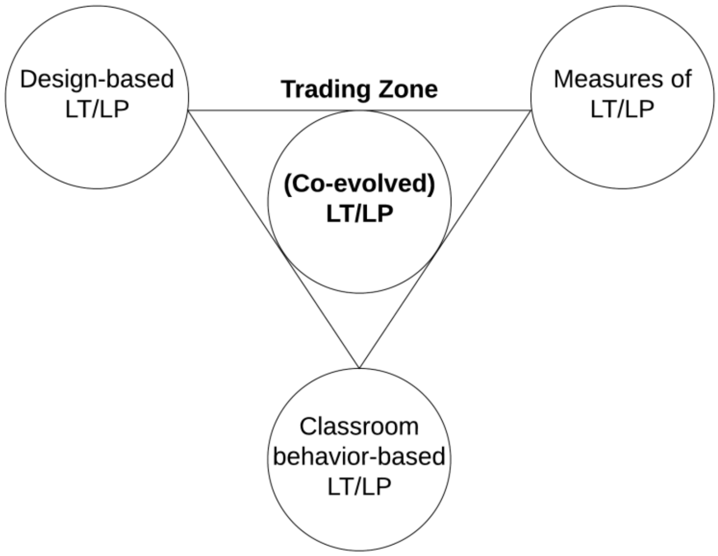
[5] Science educators also explain that the term heat is itself misleading and informal and that the more accurate distinction is between kinetic/thermal energy and temperature. I chose to use "heat" in the analogy to reach a wider, less expert audience.

variability, and chance. Partnering with Wilson, Lehrer proceeded to create construct maps for the related ideas in statistical reasoning. They used both unidimensional IRT and a multidimensional random coefficient multinomial logit model (MRCML) to model the student responses. They further engaged in creating a professional development model that involved teachers in strengthening their knowledge of the content, understanding the design of a set of lessons, and employing the assessment system to interpret student responses. This collaborative work flowed multi-directionally among the three teams--the practitioners, the learning scientists, and the measurement specialists--during 13 day-long workshops conducted across two years. Lehrer et al. (2014) report four distinct forms of formative assessment practices: using the items to gauge correctness of student performance, using the items to generate increased student participation, intentionally eliciting diverse student responses from construct maps, and teachers' efforts to compare and contrast results. Lehrer et al. (2014) refer to this deeply collaborative work among learning scientists, measurement experts, and teachers as leading to the consideration of "a learning progression[6] as a *trading zone* (sensu Galison, 1997) in which different realms of education practice intertwine, much as a cable is constructed" (p 54) (with the realms identified as learning theory, assessment, instruction and professional development).

This type of collaboration suggests a means (Figure 3) to develop and refine the use of LT/LPs in improving classroom instruction, by combining all three approaches described in previous sections. In Lehrer et al.'s (2014) study, the foundation of the work (in this case, data modelling) was solidly based on years of prior research on learning statistical reasoning. The project itself was conducted as a design study; hence it represented an example of "design-based LT/LPs." The practitioners in the collaboration were active participants in contributing to and shaping the ideas, providing feedback on materials, and testing the conjectures in a manner substantially consistent with Leahy and Wiliam's (2011) "classroom- behaviour-based LT/LPs," solidly rooted in practice. And Wilson's contributions to the "measures of LT/LPs" offered insights into the structure of the interrelationships among the LT/LPs. Thus, the three approaches to the study of LT/LPs can converge within a trading zone, to create a stronger, co-evolved LT, through which would likely be versions of "design-based implementation research" (DBIR) (Fishman, Penuel, Allen, & Cheng, 2013) or "networked improvement communities" (Bryk, Gomez, & Grunow, 2010; Russell, Bryk, Dolle, et al., 2017). This integrated approach has the advantage of reducing (or avoiding altogether) the need to conduct separate design studies and measurement initiatives. At the same time, it improves the practicality of the approach and its immediacy for and credibility with teachers.

[6] Lehrer et al. (2014) make a distinction between a learning progression - representing the larger overall system that addresses the implementation of the model of data modelling progressively through the combined activities of learning, teaching, and assessing over long periods of time - and learning trajectories - the "prospective pathways of conceptual development in outcome space defined by constructs and learning performances" (p. 53).

Figure 3. Trading zone among three approaches to LT/LP to generate a richer "co-evolved" LT/LP.



6. What Evidence is there from Taking LT/ LPs to Scale?

If the increased attention paid to learning trajectories/learning progressions is to translate into widespread positive impact on student learning, the applications of LT/LPs must be taken to scale. A few research teams are tackling this challenge. Going to scale requires the use of a large number of LT/LPs across a broad span of grade levels in multiple locations (schools, districts, states, regions and countries), but it requires more than increasing the numbers of participants. It requires one to weave all the components of the LT/LP into an integrated system that incorporates careful attention to the connections between LT/LPs and the curriculum (materials), instruction, classroom assessment (formative and measurement-based), and all forms of professional development, support, and capacity building.

Going to scale with LT/LPs in a way that authentically represents the research on LT/LPs requires that the focus of all the components be on student learning. Classrooms must become places where student ideas are driving the interactions, as the students undertake series of tasks designed to elicit their thinking and to move them progressively toward more sophisticated thinking. Teachers carry out the critical role of facilitating student learning, based on listening to student ideas, collecting and analysing all forms of evidence of their thinking and understanding, and providing them appropriate and timely opportunities to learn and tackle the next ideas and obstacles. A focus must be on building their self-awareness as learners, strengthening their agency, and inviting them into a reciprocal partnership on assessment (Adie, Willis, and Van der Kleij 2018; Fletcher, 2018).

Assessment of LT/LPs plays a central role of generating feedback in the instructional process. Emerging efforts to systematically provide that feedback through repeated measures, across all learners, and in a timely way will contribute significantly to making classroom instruction and learning a more responsive and dynamic process. While contributing to ways to become more systematic with the measures, it is essential, according to theory of LT/LPs, for the LT/LPs to be shaped by and remain sensitive to the local conditions and resources, and to be able to constantly evolve (Delgado & Morton, 2012).

In order for the LT/LPs to evolve and to be locally sensitive, the activity of learning must recognise the requirement for ongoing partnership among teachers, learning scientists, and measurement experts in a brokered “trading system” (Lehrer et al., 2014). Across the globe, we have seen, and increasingly are seeing, significant concentrated efforts to undertake the challenge of moving to scale. There is much to be learned from these initial efforts.

Clements, Sarama and colleagues (Clements & Sarama, 2011; Sarama, Clements, Wolfe, & Spitler, 2012; Clements, Sarama, Wolfe, & Spitler, 2013) have reported extensively on a TRIAD (Technology-Enhanced, Research-Based Instruction, Assessment and Professional Development) study of 42 schools located in two urban low-resource communities. Teachers used their “Building Blocks" curriculum (Clements & Sarama, 2007) and accessed their web application “Building Blocks Learning Trajectories” with a variety of videos explaining examples of student work. Primary grade teachers were provided 12 days of professional development as they implemented the curriculum. Students in the experimental group scored significantly higher than those in the control group, and African-American students showed increased benefits in the experimental compared to the control group.

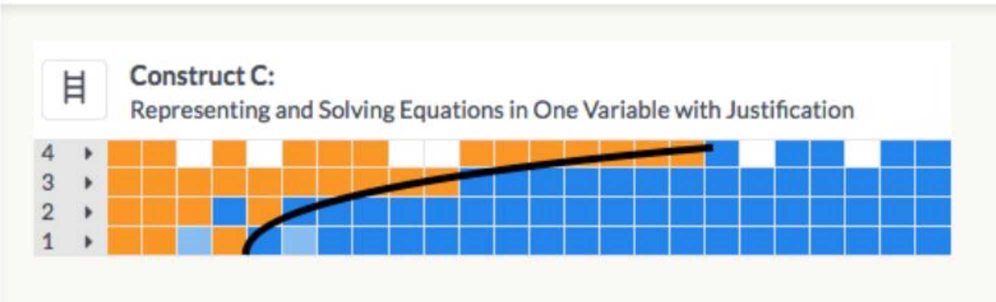
Lehrer and colleagues (Lehrer, Jones, Pfaff & Shinohara, 2017) in a decade-long research program on supporting the development of student reasoning about variability by introducing students to statistical practices of data modelling, identified and built learning trajectories for five sub-constructs: data representation/visualization (DaD), conceptions of statistics (CoS), conceptions of chance (Cha), understandings of modelling variability (MoV), and conceptions of informal inference (InI). Measures of student learning were designed and validated for each sub-construct (Lehrer et al, 2014). The research culminated in an eight-week design experiment involving middle grades students and teachers, studying the ways and degrees to which students learned to engage in and adopt the practices of data modelling. A randomised cluster approach was implemented across 20 schools in four districts, matched with control schools. Hierarchical linear modelling and item response modelling of multiple measures of student learning, including those developed previously by the researchers, supported the efficacy of data modelling for promoting the development of statistical reasoning in the early middle school years, showing a moderate effect size in the difference between experimental and control groups. In related sub-studies, they also demonstrated that the degree to which teachers adopted the methods of having students invent and compare representations, measures, and models predicted student achievement. Forms of teacher implementation were also traced from professional development settings where teachers rehearsed discourse moves that would sustain student engagement and promote statistical reasoning to teacher use of these forms of discourse in their classrooms. Using case studies, they reported on specific students' epistemic practices and resulting agency, discussed in a previous section (Shinohara & Lehrer, 2018). This canon of work demonstrates the challenge of a full research programme on learning trajectories. Developing design-based LT/LPs alone requires multi-year studies of the evolution of student thinking; such work is accompanied by and extended by years more to design and revise relevant curricular materials, (especially in novel topics for a grade level such as data modelling), and create and validate measures of LT/LPs. Only then can the work be taken to classrooms at scale, providing curricular materials and associated professional development, with evidence from multi-method studies. The work at scale often provokes new challenges for the LT/LP. For instance, the need for explicit rehearsal of forms of thought-provoking discourse during professional development was inspired by the challenges of the need for rapid induction of teachers into data modelling practices that were posed by the scale of the design experiment.

At the upper secondary level, Siemon, Callingham, Day, Horne, Seah, Stephens, and Watson (2018) conducted a three-year study of the development and implementation of evidence-based learning progressions for algebraic reasoning (Day, Stephens, & Horne, 2017), spatial reasoning (Horne & Seah, 2017), and statistical reasoning (Watson & Callingham, 2017). The project involved 80 teachers and 3500 students in grades 7-10. Conducted in three phases, the project's goals were to develop tasks, design scoring rubrics and collect initial student data to conduct Rasch analysis for the purpose of articulating the LPs. Year two goals included the development of multiple forms of assessment, the conduct of teacher surveys, and the development of instructional materials for targeted teaching (Siemon, 2017). Reporting specifically on spatial reasoning, Siemon, Horne, Clements, et al. (2017) described a learning progression with seven zones defined by setting cut scores within a Rasch analysis. Each zone was accompanied by teaching implications (no report was offered on the effects of the use of tool on student or teacher learning.) The authors postulate that the LPs support teachers in seeing the big ideas within the Australian Curriculum: Mathematics (ACM), and in making more informed curricular decisions.

Going to scale also can mean leveraging digital technology extensively to support the use of learning trajectories across grades and topics and to return assessment data in real time to students and teachers. Confrey, Gianopulos, McGowan, Belcher and Shah (2017) report on their creation and use of a digital learning system (DLS), "Math-Mapper 6-8," (MM6-8) at three middle grades schools in two districts. The system was designed to help teachers respond precisely and rapidly to classroom diagnostic assessments built to measure progress along LT/LPs. The assessments are administered immediately after initial instruction in a topic, and based on the demonstrated needs from the data, teachers personalise their subsequent instruction. Retesting and practice are made available to students to gauge subsequent progress along the LT/LPs.

To achieve the designer's goal to focus the teachers on learning targets instead of standards, the system was designed with a map of nine big ideas organised hierarchically into 23 clusters made from 62 learning trajectories. Each LT is linked to the related Common Core State Standard for Mathematics (U. S.). Progress along learning trajectories is measured at the level of a cluster in the map using a diagnostic assessment (digitally delivered and scored, 30-minute duration, multiple forms and different-grade level tests). Reports are immediately digitally returned to both students and teachers in the form of "heat maps," with the levels shown from bottom to top and the students ordered from lowest- to highest-performing on each construct (from left to right). The cells are colour-coded showing scores from orange (incorrect) to blue (correct). The sweeping Guttman curve quickly informs teachers which students and which levels need more attention (see Figure 4).

Figure 4. A heat map for a LT/LP with the levels displayed vertically and students ordered horizontally from lowest to highest performing on the measure. Orange indicates incorrect responses and blue correct ones.



Confrey et al. (2017) report on the results of the diagnostic assessments in algebra with MM6-8 with three schools in two districts serving more than 2000 students. Simple regression analyses empirically recovered the LT showing lower mean correct proportions per item for ascending progress levels for each of the four LTs (describing patterns and relations using algebraic expressions, translating, substituting and finding equivalent expressions, and representing and solving equations and inequalities in one variable). Confrey et al. (2019) report on how an ongoing process of validation that involves a deep collaboration among learning scientists, psychometricians, and practitioners is required to support progress in using the DLS at scale. Those collaborations include innovations in applying psychometric models, revising and modifying LTs based on empirical data and discussions with practitioners, and adding features to the tool to respond to feedback from users.

Other efforts to go to scale with LT/LPs include Battista's (2011), ETS's CBAL (Bennett 2011), and Petit's (2011) OGAP system. Experience with these initial efforts to take LT/LPs to scale show promise in strengthening student learning and also emphasise the need to provide sufficient professional development, involve administrators in the planning and implementation, and communicate well with all parties. They also point to the need for ongoing processes of validation of assessment instruments and careful designs to tie properly to the larger assessment system (Shepard et al., 2018a).

6.1. Types of Outcomes from LT/LP studies

After reviewing the research on LT/LPs in mathematics, one might ask: Can one conclude that LT/LPs have a positive impact on student outcomes? Throughout this review answers have been presented, which are summarised and contrasted here. First, researchers conducting design studies demonstrate that students, indeed, do display the behaviours or ways of thinking associated with the constructs in the levels, so that the delineation of the trajectory with examples is itself an outcome of the research. A second question of outcomes might concern the prevalence of certain behaviours or ways of thinking associated with levels. Some research studies on LT/LPs employ cross-sectional cross-grade designs to answer such questions, which provides a second meaning of outcomes. Building these studies on previous design studies provides some insights in the frequencies of students' positioning at different levels of performance, but such approaches are: a) only as valid as the measures they use, b) measure the frequency of outcomes within typical practice, and c) will lack descriptions of the "epistemic practices" and mechanisms describing or explaining students' movement between levels. Nonetheless, the cross-sectional studies add to the body of the literature of outcomes in LT/LPs. A third means to describe the outcomes of LT/LPs is to validate the measure of LT/LPs by applying various measurement models to data from student performance on items designed to measure the constructs for the levels. Occasionally, measurement-based LT/LP studies involve correlations to other measures as a form of criterion-validity; more recently, validation is situated within a larger validation argument including some form of attention to consequential validity (Kane, 2013; Ketterlin-Geller et al., 2018; Carney and Smith, in press; Confrey, et al., 2019).

A fourth answer to the question of what constitutes the outcomes of LT/LP research comes from the programme taken to scale. Few have reached this level of sophistication as these rest on the foundation of all of the other genres of LT/LP research. Furthermore, demonstrating effects at scale is challenging due to the complexity of the logic model presented in Figure 1. LT/LPs are deeply embedded in the teaching-learning setting, almost akin to the role of fascia in human anatomy, sheets of connective tissue that attach, separate, and stabilise muscles and internal organs, imperceptible to most people until they become inflexible or inflamed. One assumes that LT/LPs are operating to some degree all the time, but that by identifying and building them, by detailing their meaning and articulating the ways they can emerge and foster learning of the content, one can strengthen learning and assist students in moving along different interwoven pathways. Thus, work at scale requires this network of connective tissue to be activated more robustly, and studies at scale therefore require the whole system to light up, from engagement in curricular tasks, to more lively instructional exchanges, to fostering shared responsibilities and agency in formative assessment practices and subsequently to performance on student learning outcome measures. The approach demands systemic change, so demonstrating positive student learning outcomes will be a gradual process requiring widespread commitments and broad policy supports.

Unclassified

7. What is Known about LT/LPs' Impact on Educational Policy?

Learning trajectories' effects on educational policy are emerging from around the world. In large part, the success of these efforts will depend on the practical speed at which the research can be taken to scale, but much can also be learned as the approach moves to the national level. Primary examples are seen in the United States and Australia.

In the United States, research on student learning, specifically LT/LPs, was reported as a major input to the development of the "Common Core State Standards in Mathematics" (CCSS-M). The CCSS-M writers acknowledged this in the introduction by writing,

> *In addition, the "sequence of topics and performances" that is outlined in a body of mathematical standards must also respect what is known about how students learn. As Confrey (2007) points out, developing "sequenced obstacles and challenges for students...absent the insights about meaning that derive from a careful study of learning, would be unfortunate and unwise." In recognition of this, the development of these Standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (CCSSI 2010, p.4)*

Daro, Mosher, and Corcoran (2011) reflected on the relationship between standards and LT/LPs:

> *Decisions about sequence in standards must balance the pull of three important dimensions of progression: cognitive development, mathematical coherence, and the pragmatics of instructional systems. (p. 41)*

Once CCSS-M was released, various scholars endeavoured to generate more detail on the relationship between the Standards and LT/LPs. Some critiqued the adequacy of the standards' considerations of underlying LT/LPs (Smith & Gonulates, 2011), while others saw the standards as scaffolding that would benefit from a more complete articulation of LT/LPs (Confrey, Nguyen, Lee, et al., 2011; Common Core Standards Writing Team, 2011-2018; Hess, 2018). From these efforts, some contrasts and clarifications can be drawn regarding the relationships between curriculum [standards] and LT/LPs.

Firstly, standards represent negotiated agreements regarding when, during schooling, a topic should be learned. This tends to represent to most educators when a particular standard can be assessed. But if the ramp to understanding for that standard exceeds a year, then standards are silent on when to begin teaching that topic. LT/LPs can make this contribution.

Secondly, an analysis of CCSS-M showed significant variation in the grain size of standards, so that using them to guide curriculum planning could be misleading. LT/LPs can mediate the problem of widely varying grain sizes of curriculum [standards]. LT/LPs, based on far more uniform grain size and their specific learner-centred emphasis on the way student understanding progresses and evolves, can fill in the needed detail for instructional planning.

Thirdly, based on examples in CCSS-M, the construction of individual curriculum [standards] may be guided more by the logical categories of the disciplines than by the underlying learning issues. For example, the 6th grade Statistics and Probability standard 6.SP.B.4 ("Display numerical data in plots on a number line, including dot plots, histograms, and box plots") seems to imply close relationship in the sequencing and cognitive demand of three disparate representations (dot plots, histograms and box plots) but in fact, learning research clearly demonstrates that box plots require very different student reasoning experience and sophistication about measures and distributions. Using this standard as a single instructional target would likely impede coherent instruction and student understanding.

These analyses suggest that even when LT/LPs are considered in the preparation of curriculum [standards], the differences between the purposes of curriculum [standards] and the purposes of LT/LPs have to be considered. The relationship should be complementary, with the two components recognised as distinct.

Australia is launching an even more ambitious effort to drive school improvement with LT/LPs. A recent Australian report, "Through Growth to Achievement: Report of the Review to Achieve Educational Excellence in Australian Schools," also known as the "Gonski Report" (Commonwealth of Australia, 2018), focuses on student learning as measured by growth (Priority One: "Deliver at least one year's growth in learning for every student every year"). The report specifies a key role for learning progressions in achieving growth:

> *To achieve this shift to growth, the Review Panel believes it is essential to move from a year-based curriculum to a curriculum expressed as learning progressions independent of year or age. Underpinning this, teachers must be given practical support by creating an online, formative assessment tool to help diagnose a student's current level of knowledge, skill and understanding, to identify the next steps in learning to achieve the next stage in growth, and to track student progress over time against a typical development trajectory. (Executive Summary, p. X)*

Of the report's 23 recommendations, learning progressions play a significant role in four (nos. 5, 6, 7, 11), including use in curriculum [standards], delivery, and formative assessment. Furthermore,

> *All Australian education ministers agreed to collaborative action to develop national literacy and numeracy learning progressions in December 2015. Since then, learning progressions in literacy and numeracy have been developed for use in some states and territories. (p. 33)*

The Australian policy work is heavily informed by the work of the Australian Council for Educational Research Center for Global Education Monitoring (ACER-GEM). Representing ACER-GEM, Adams et al. (2018) seek to inform the development of the next round of international indicators of educational quality, central to goal 4 (Quality Education) of the United Nations 2030 Agenda for Sustainable Development (United Nations, 2015). They acknowledge that, in defining quality across various countries, an inherent tension has often risen between measuring quality based on a shared international test and risking excessive standardisation and influence by dominant cultures. They label the use of a standardised test a rigid but comparative solution on the one hand, and the approach to allow each locality to determine their own standards and definitions a flexible but idiographic solution, on the other. They believe it is possible to build a set of learning progressions that "describe a construct independently of any particular assessment tool used to measure it… Although different kinds of rulers may be used to measure length, these measurements are consistent because of the common understanding of length that informs their design." They envision learning progressions as helping to provide countries with empirically-based information about how LT/LPs can help “improve the quality of their curricula, teaching and learning, school resources, and assessment programs.” (p 3.)

Adams et al. 2018 describe learning progressions as “a scale that defines the constructs that constitute educational progress in a particular domain (say, reading or mathematics).” (p. 2). They elaborate further:

> *Learning progressions are directional, in that lower points on the scale represent less learning, and higher points represent more. Locations along the scale may be described numerically, as proficiency scores, or substantively, as proficiency descriptions. The proficiency descriptions make it clear what learners are expected to know and be able to do at designated levels on the scale, while the proficiency scores enable learning to be quantified against the scale.*

Defining a learning progression as a scale risks paying inadequate attention to the qualitative character of the levels of the underlying design-based LT/LPs, as discussed in Section 5.3. However, later in the brief the authors seem cognisant of the importance of emphasising the constructs in the levels, arguing that the greatest benefit of LT/LPs will be their focus on learning constructs rather than just on test scores (p. 4). They also call for an “extensive consultation with members of the international education community, including leaders in cross-national assessments, learning domain and curriculum experts, and national curriculum, assessment and education policy teams from the widest possible range of countries.” Including “learning domain and curriculum experts” in the list of consultants, if used robustly throughout the process of design, implementation and evaluation, can allow them to create a “trading zone” (section 5.4).

These two country examples of policy initiatives related to LT/LPs are offered as illustrative exemplars of the use of LT/LPs and as indicators of their increasing importance.

8. What are the Possible Future Roles of LT/LPs in the OECD's 2030 Vision and Learning Framework?

The OECD Education 2030 project offers a perspective on the knowledge, skills, attitudes and values needed by students in the coming dozen years, and strives to anticipate the instructional systems and professional educators' capacities necessary to achieve that vision. The associated Learning Framework communicates an ambitious and urgent message. Children entering the school pipeline in 2018 will encounter fundamental shifts in the world environment, economy, and socio-cultural conditions and context, including scarcely imagined opportunities from new discoveries, and severe challenges from political/social upheavals and limitations in resources. The Project 2030's learning framework acknowledges these changing conditions and anticipates the need to help students be ready to face them with a strong foundation in academic knowledge and dispositions of preparedness for change. Constantly exposed to a world that can be both inspiring and brutal, both encouraging and deceptive, our next generation of students must be resilient, persistent, self-aware, self-motivated, cooperative, and determined. In order to meet the challenges of this vision, educators are charged to design educational systems that are carefully and expertly built on the most up-to-date and informative insights about student understanding and learning. Furthermore, these systems must anticipate continued change: they must be designed for iterative improvement based on data and feedback from ongoing educational practice.

Educational goals are typically expressed as lists of competencies and skills students must attain. As knowledge has accrued, those lists have ballooned, and the job of teaching has become far more difficult. Learners are saddled with unrealistic expectations that often fail to represent the reality of a world in which sources of information - and *mis*information - abound, and search engines are ubiquitous and increasingly responsive to detailed queries. The "signal" in these complex systems - the knowledge that undergirds ideas with broad explanatory power - can be too easily lost in the fragmented snippets and twitter-based noise that distracts, cycles rapidly, and clamours for attention. In order for our students to achieve the vision of the OECD 2030 learning framework, we must focus on teaching them big ideas - ideas that connect many examples and support students in generating traction for explanations of a broad range of phenomena.

Research on learning has revealed a number of domain-specific insights into how students learn big ideas as they progress from holding naive yet intuitive nascent ideas through levels of increasing sophistication. Descriptions of these patterns of evolving reasoning have been called "learning trajectories" or "learning progressions." LT/LPs are not stage theories; they depend on providing students opportunities to undertaken challenging tasks, participate in active and engaged discussions, and make use of a variety of tools and representations. In a phrase: instruction plays a prominent role in the process. However, findings from relevant studies of learning typically lack synthesis and systematisation, and are often dispersed throughout the literature, so communicating them widely to educational practitioners is a serious challenge.

A second theme that resonates between the Learning Framework and the research on LT/LPs is **the recognition that robust learning requires an active and aware learner with a sense of agency**. The argument for this is twofold. One, for student knowledge to become sophisticated and generative, the learner must her or himself become aware of the process of continuously refining or reforming her or his own understanding. Students must become partners with teachers and with each other in recognising that knowledge is not simply accretion or assimilation, but rather involves episodes of transformation, evaluation and choice, or as Piaget explained, *accommodation* (Piaget, 1976). Secondly, the learner's role involves the negotiation of meaning in social contexts involving purpose, interest, and responsibility. Deep learning over time requires active participation by students. Students must come to understand that knowing and learning are continual, ongoing processes, and that paths to expertise *are* accessible through careful and persistent study. A focus on learning trajectories/progressions initiates students' participation in a process of systematic learning, not merely the distant hope of successfully obtaining an endpoint.

Based on this review of the literature, it seems clear that scholars are synthesising many of the empirical insights into student learning into sequences of evidence about the landmarks and obstacles that students encounter as they move from naive to sophisticated understanding of big ideas. These sequences can have a variety of timescales - over days, weeks and months, not only over years. LT/LPs, especially at finer grain sizes, have the potential to be highly informative to teachers as they conduct instruction, because the LT/LPs describe both the emergence of students' nascent ideas and paths to the horizon for which they may be headed.

This paper began with a description of the teacher's dilemma of addressing students' learning needs vs. maintaining the grade-level expectations. By specifying the probabilistic paths of learning, the knowledge base connected with learning trajectories provides a possible direction for resolution. Instead of labelling "lagging students" as unaligned with the required standards (a deficit perspective) and losing track of them, students at differing levels of performance can be viewed in terms of what they are able to do on the related pathway and evaluated for their progress along the path. Further, a learning trajectory approach, in contrast with a purely developmental approach or stage theory, recognises the critical role for instruction and thus places the onus on the system - thus, there is a clear recognition that the student needs the educational system to afford her or him a particular instructional opportunity. Instead of merely "gap gazing" (Gutiérrez, 2008), the educational system should provide a set of diagnostic indicators that provide concrete instructional suggestions and resources.

This proposed system will *not* be realised with another international test, though such tests may nonetheless retain value as summative comparative tools. What this calls for instead is the design of a dynamic feedback and learning system, implemented as part and parcel of instruction, with time intentionally and regularly devoted to acting on the results, in order to meet the individual needs of students. It requires a classroom assessment system with strong formative goals and practices. To be applicable to the broadest set of cases, that system will need to leverage a variety of technologies, including the means to assist teachers to work directly with samples of student work, and accessible tools and materials to address what is learned about student progress.

8.1. Considerations

As with most innovative ideas designed to address pressing and pernicious problems, it is important to be explicit and to hold persistently to certain understandings and principles about the key idea of LT/LP based on the results of this synthesis. To conclude, a set of considerations is proposed, following from the research syntheses presented in this article.

1. **Hold to the core ideas about design-based LT/LPs.** Each LT/LP (a) should be developed through the conduct of empirical qualitative studies of the evolution of student thinking over time, using rich tasks to elicit a broad range of student ideas (a methodology known as "design study"), and (b) should be accompanied by an explicit description of its underlying theoretical assumptions about constructivist learning as situated in socio-cultural contexts. It is important to recognise that LT/LPs derive from *qualitative empirical* studies of learners; they are not created merely by scholarly attempts to deconstruct mathematical ideas into their logical subparts. Logical deconstruction is a process that can inform the initial conjecture about an LT/LP, but should not be mistaken for an LT/LP.
2. **Launch a systematic international effort to develop learning trajectories for under-researched and emerging topics.** Expect existing LT/LPs to evolve with the introduction of important emerging topics such as such as computational reasoning (Rich, Binkowski, Stickland & Franklin, 2018), new findings in the learning sciences, and new representations and contexts.
3. **Conduct cross-cultural research on LT/LPs to investigate their sensitivity to differences in context, language, representation, and instructional practice.** Conceptualisation of LT/LPs involves student beliefs, experience, language, representations, and instructional experiences, so cultural differences in LT/LPs should be expected (Delgado & Morton, 2012). On the other hand, despite the fact that mathematics has evolved in diverse settings (times and places), it has produced many common ideas: a high degree of generality of ideas is likely to be seen in cross-cultural studies of LT/LPs. Thus, cross-cultural studies of LT/LPs should be undertaken with an open mind about possible commonalities and differences among contextual results.
4. **Distinguish the institutional/organisational role of curriculum [standards] from the empirical research-based character of LT/LPs, and carefully coordinate their use.** Curriculum [standards], developed through negotiated agreements among experts in mathematics and mathematics education, provides *organisational guidance* about what to teach and when to teach it. LT/LPs, developed by learning scientists in mathematics, provide detailed, empirically-supported information regarding *documented patterns in students' learning of the content* that is indicated by the curriculum [standards].
5. **Document and/or measure student progress on LT/LPs, to provide valid, systematic, and timely feedback for improving ongoing instruction and learning.** Varied degrees of technology can be leveraged to provide feedback on student progress along LT/LPs, ranging from means to share artefacts of student work (such as document cameras) to the use of dynamic digital learning and assessment systems that return analysed data in real time. In the international context, rapid progress requires careful consideration of available technological resources and the related human capacity for training and use.
6. **Distinguish a LT/LP from the measures of an LT/LP, and research both by applying appropriate theory and method.** LT/LPs model how students think. They are also designed to anticipate and capture unexpected responses. Measures typically create scales or categories to measure or classify a students' progress along an LT/LP. It is critical to ensure that measures are adequately grounded in relevant research on learning, i.e. in relation to design-based LT/LPs. Recognise

also that elements of the measurement process and its validation will contribute further theoretical and empirical insights into LT/LPs.

7. **Promote students' active participation in monitoring their progress on LT/LPs to build self-regulation and agency.** The precisely-specified levels of LT/LPs supports students' participation in self-regulated learning (SRL) as they "generate, monitor, and adapt thoughts, behaviours, and feelings in pursuit of goals" (Fletcher, 2018, p.2). Extending SRL to include student agency as "the intentional planned pursuit of goals and the initiation of appropriate action to reach an anticipated outcome" (Bandura, 2006, p. 2), teachers and students can participate together in formative assessment practices as a reciprocal process (Fletcher, 2018) designed to help students obtain the full range of goals of schooling: employment, informed citizenry, personal development, and competence (Klemenčič, 2015).
8. **Design and implement learning organisations and related technological systems around LT/LPs based on deep and ongoing collaborations among learning scientists, measurement specialists, and expert practitioners.** These innovative organisational configurations would be designed to support adaptive and responsive LT/LP-based instruction at scale, to develop, test and revise new scientific discoveries about LT/LPs and their measures, and to inform the gradual revision of long-term curricula [standards] and materials. Success would be measured in the impact of the system on student learning outcomes.

These eight considerations, drawn from a synthesis and interpretation of the existing literature on LT/LPs, are starting points for a rich discussion among member nations. Consideration 1 ensures an adequate foundation of LT/LPs in the learning sciences, and places students as the centre of the process. Considerations 2 and 3 advocate for a comprehensive treatment of LT/LPs, and recognise the importance of investigating them in diverse cultural settings.

Consideration 4 clarifies the relationship between LT/LPs and curriculum [standards], and thus promotes reconsideration of the original dilemma, posed at the outset of the paper, between learning needs and grade-level expectations. Distinguishing LT/LPs and curriculum [standards] allows one to distinguish two different purposes of assessment: measuring an attainment of curriculum [standards] at grade level (compliance) vs. examining students' progress along LT/LPs using artefacts of student work or measures of LT/LPs (diagnosis and guidance). The resolution of the dilemma rests in recognising the value of the two kinds of assessment targeting to different audiences (policy makers vs. school practitioners) and across different timelines (annual vs. proximal to actual instruction).

Considerations 5 and 6 recognise the value of LT/LPs in providing systematic, efficient, and comprehensive feedback to students *and* teachers during instruction and advises on how to ensure that measurement of LT/LPs is grounded appropriately in the learning sciences. The importance of partnering with students in assessment practices *for* learning and to strengthen student agency is emphasised in consideration 7. Finally, consideration 8 offers a vision for creating a dynamic digital learning and assessment system, one that uses LT/LPs and their measures to continuously leverage data on students' specific needs, to improve instruction through effective collaborations among learning scientists, measurement specialists, and practitioners. The field of research on LT/LPs is still relatively young and emerging, but its potential to inform the next steps needed to improve and support learning by all students at scale is promising and worthwhile.

Appendix A. A List of Learning Trajectories/ Progressions in Mathematics by Strand, Topic, and Grade Level

Content Strand	Topic	Grade Level	Author(s)
ALGEBRA AND FUNCTIONS	Equations	Middle school	Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning,* 9(3), 221-247.
	Equality and Variable	Middle school	Arieli-Attali, M., Wylie, E. C., & Bauer, M. I. (2012). The use of three learning progressions in supporting formative assessment in middle school mathematics. Presented to the annual meeting of the American Educational Research Association, Vancouver, Canada.
	Functions	Middle and High school	Ayalon, M., Watson, A., & Lerman, S. (2015). Progression towards functions: Students' performance on three tasks about variables from grades 7 to 12. *International Journal of Science and Mathematics Education*, 1–21. doi:10.1007 /s10763-014-9611-4
	Functions	Elementary school	Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year- olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, 46(5), 511–558. doi:10.5951/jresematheduc.46.5.0511
	Linear functions	Middle school	Chiu, M. M., Kessel, C., Moschkovich, J., & Muñoz-Nuñez, A. (2001). Learning to graph linear functions: A case study of conceptual change. *Cognition and Instruction,* 19(2), 215-252.
	Linear and Quadratic Functions	Middle school	Ellis, A. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A Global Dialogue from Multiple Perspectives* (pp. 215-235). New York: Springer.
	Exponential Functions	Middle school	Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., Williams, C., & Amidon, J. (2013). Correspondence and Covariation: Quantities changing together. In Martinez, M. & Castro Superfine, A (Eds.). (2013). Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Chicago, IL: University of Illinois at Chicago. Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39*, 135–155. doi:10.1016/j.jmathb.2015.06.004

	Linear Functions	Middle school	Graf, E. A., & Arieli-Attali, M. (2015). Designing and developing assessments of complex thinking in mathematics for the middle grades. *Theory Into Practice*, *54*(3), 195-202. Arieli-Attali, M., Wylie, E. C., & Bauer, M. I. (2012). The use of three learning progressions in supporting formative assessment in middle school mathematics. Presented to the annual meeting of the American Educational Research Association, Vancouver, Canada.
	Quadratic Functions	High school	Graf, E. A., Fife, J. H., Howell, H., & Marquez, E. The Development of a Quadratic Functions Learning Progression and Associated Task Shells. *ETS Research Report Series*.
	Algebra	Elementary and Middle school	Ketterlin-Geller, L.R., Shivraj, P., Basaraba, D., & Schielack, J. (2018). Universal Screening for Algebra Readiness in Middle School: Why, What, and Does It Work? *Investigations in Mathematics Learning*.
	Trigonometry	High school	Moore, K. C. (2010). The role of quantitative reasoning in precalculus students learning central concepts of trigonometry. Arizona State University. Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context, 2, 75-92.
	Linear Functions and Proportional Reasoning	Middle school	Pham, D., Bauer, M., Wylie, C., & Wells, C. (under review) Using cognitive diagnosis models to evaluate a learning progression theory.
	Early Algebra	Elementary school	Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning,* 19(3), 143-166.
	Beginning Algebra	Middle school	Tabach, M., Hershkowitz, R., & Dreyfus, T. (2012). Learning beginning algebra in a computer-intensive environment. *ZDM* 45(3) 377-391.
	Two-variable functions	High school	Weber, E., & Thompson, P. W. (2014). Students' images of two- variable functions and their graphs. *Educational Studies in Mathematics*, 87, 67–85. doi:10.1007/s10649-014-9548-0
	Functions	Middle school/High school	Wilmot, D. B., Schoenfeld, A., Wilson, M., Champney, D., & Zahner, W., 2011. Validating a learning progression in mathematical functions for college readiness. *Mathematical Thinking and Learning* 13(4) 259-291.
	Functions	Middle school	Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education*, 431-466.

GEOMETRY	Geometric figures	Elementary school	Clements, D. H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning,* 6(2), 163-184.
	Space and Geometry	Middle school	Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. *Journal for Research in Mathematics Education*, 46(4), 423–454. doi:10.5951/jresematheduc.46.4.0423
	Space and Geometry	Elementary school	Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children’s reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 137-167). Lawrence Erlbaum Associates, Mahwah, NJ.
	Similarity	Elementary school	Lehrer, R., Strom, D., & Confrey, J. (2002). Grounding metaphors and inscriptional resonance: Children's emerging understanding of mathematical similarity. *Cognition and Instruction*, 20(3), 359-398.
	Similarity and Scaling	Middle school	Shah, M. J. (2018). Applying a Validity Argument Framework to Learning Trajectories on Middle Grades Geometric Similarity Using Learning Science and Psychometric Lenses. *Unpublished dissertation*.
	Spatial Reasoning	Elementary and Middle school	Siemon, D., Callingham, R., Day, L., Horne, M., Seah, R., Stephens, M., & Watson, J. (2018). From research to practice: The case of mathematical reasoning. *MERGA 41: Annual conference of the Mathematics Education Research Group of Australasia*. Siemon, D. & Callingham, R. (2018). Researching Mathematical Reasoning: Building Evidence-based Resources to Support Targeted Teaching in the Middle Years. In D. Siemon, T. Barkatsas & R. Seah (Eds.), *Researching and using learning progressions (trajectories) in mathematics education.* Leidan, the Netherlands: SENSE Publishers.
MEASUREMENT	Length Measurement	Elementary school	Barrett, J. E., & Clements, D. H. (2003). Quantifying path length: Fourth-grade children’s developing abstractions for linear measurement. *Cognition and Instruction*, 21(4), 475–520. doi:10.1207/s1532690xci2104_4 Barrett, J. E., Clements, D. H., Klanderman, D., Pennisi, S. J.,& Polaki, M. V. (2006). Students’ coordination of geometric reasoning and measuring strategies on a fixed perimeter task: Developing mathematical understanding of linear measurement. *Journal for Research in Mathematics Education*, 37(3), 187–221. doi:10.2307/30035058 Barrett, J. E., Sarama, J., Clements, D. H., Cullen, C., McCool, J., Witkowski-Rumsey, C., & Klanderman, D. (2012). Evaluating and improving a learning trajectory for linear measurement in elementary grades 2 and 3: A longitudinal study. *Mathematical Thinking and Learning*, 14(1), 28-54. Sarama, J., Clements, D. H., Barrett, J., Van Dine, D. W., & McDonel, J. S. (2011). Evaluation of a learning trajectory for length in the early years. *ZDM*, 43(5), 667. Szilágyi, J., Clements, D. H., & Sarama, J. (2013). Young children's understandings of length measurement: Evaluating a learning trajectory. *Journal for Research in Mathematics Education*, 44(3), 581-620.
	Length	Elementary school	Battista, M. T. (2006). Understanding the Development of Students' Thinking about Length. *Teaching Children Mathematics* 13(3), 140-146.

	Length measurement	Elementary school	Battista, M. T. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. The Mathematics Enthusiast, 8(3), 507–570.
	Area and Volume	Elementary school	Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education,* 27(3), 258–292. doi:10.2307/749365 Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education*, 29(5), 503–532. doi:10.2307/749731 Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, 30(4), 417–448. doi:10.2307/749708 Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning,* 6(2), 185–204. doi:10.1207 /s15327833mt10602_6 Battista, M. T. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. *The Mathematics Enthusiast,* 8(3), 507–570.
	Area and Circumference of Circles	Middle school	Confrey, J. & Toutkoushian, E. (2018) Middle-grades learning trajectories within a digital learning system applied to the "Measurement of Characteristics of Circles." In J. Bostic, E. Krupa, and J. Shih (Eds), *Quantitative measures of mathematical knowledge: Researching instruments and perspectives*. New York: Routledge. Refereed.
	Linear measurement	Elementary school	Gravemeijer, K., Bowers, J., & Stephan, M. (2003). A hypothetical learning trajectory on measurement and flexible arithmetic. In M. Stephan, J. Bowers, P. Cobb, & K. Gravemeijer (Eds.), *Supporting students' development of measurement conceptions: Analyzing students' learning in social context* (pp. 51–66). Journal for Research in Mathematics Education monograph series (Vol. 12). Reston, VA: National Council of Teachers of Mathematics. doi:10.2307/30037721
	Area measurement	Elementary school	Lai, E. R., Kobrin, J. L., DiCerbo, K. E., & Holland, L. R. (2017). Tracing the assessment triangle with learning progression-aligned assessments in mathematics. *Measurement: Interdisciplinary Research and Perspectives*, 15(3-4), 143-162. 10.1080/15366367.2017.1388113
	Angle Concepts	Elementary and Middle school	Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, 41(3), 209–238. doi:10.1023/A:1003927811079
	Rectangular area measurement	Elementary school	Outhred, L. N. & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education* 31(2) 144-167.

NUMBER	Proportional Reasoning	Middle school	Arieli-Attali, M., Wylie, E. C., & Bauer, M. I. (2012). The use of three learning progressions in supporting formative assessment in middle school mathematics. In annual meeting of the American Educational Research Association, Vancouver, Canada.
	Integers	Elementary school	Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education* 45(1), 19-61.
	Proportional Reasoning	Middle school	Carney, M. B., Smith, E., Hughes, G. R., Brendefur, J. L., & Crawford, A. (2016). Influence of proportional number relationships on item accessibility and students' strategies. *Mathematics Education Research Journal*, 28(4), 503-522.
	Equipartitioning	Elementary school	Confrey, J., Maloney, A., Nguyen, K. H., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education (Vol. 1). Thessaloniki, Greece: PME. Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. In K. Gomez, L. Lyons, & J. Radinsky (Eds.), Proceedings of the 9th International Conference of the Learning Sciences (Vol. 1, pp. 968–975). Chicago, IL: International Society of the Learning Sciences. Confrey, J., Maloney, A. P., Nguyen, K. H, & Rupp, A. A. (2014). Equipartitioning, a foundation for rational number reasonng: Elucidation of a learning trajectory. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), Learning over time: Learning trajectories in mathematics education (pp. 61–96). Charlotte, NC: Information Age.
	Percents	Middle school	Confrey, J., McGowan, W., Shah, M, Belcher, M., Hennessey, M., and Maloney, A. (in press). Using digital diagnostic classroom assessments based on learning trajectories to drive instruction and deepen teacher knowledge. In D. Siemon, T. Barkatsas, and R. Seah (Eds.): *Researching and using learning progressions (trajectories) in mathematics education*. Rotterdam: Sense Publishers. International, Refereed.
	Percents	Middle school	Confrey, J., Toutkoushian, E. P., Shah, M. P. (in press). A validation argument from soup to nuts: Assessing progress on learning trajectories for middle school mathematics. *Applied Measurement in Education.*
	Comparing and ordering rational numbers	Middle and High school, College	Delgado, C., Stevens, S. Y., Shin, N., Yunker, M., & Krajcik, J. (2007). The development of students' conceptions of size. In Annual Meeting of the National Association for Research in Science Teaching, April 2007. New Orleans, LA.
	Fractions	Elementary school	Hunt, J. H., Westenskow, A., Silva, J., & Welch-Ptak, J. (2016). Levels of participatory conception of fractional quantity along a purposefully sequenced series of equal sharing tasks: Stu's trajectory. The *Journal of Mathematical Behavior*, 41, 45–67. doi:10.1016/j.jmathb.2015.11.004
	Rational Number	Elementary and Middle school	Ketterlin-Geller, L.R., Shivraj, P., Basaraba, D., & Yovanoff, P. (in press). Using mathematical learning progressions to design diagnostic assessments. *Measurement: Interdisciplinary Research and Perspectives*.

	Multiplication	Elementary school	Petit, M. (2011). Learning trajectories and adaptive instruction meet the realities of practice. In P. Daro, F. A. Mosher, & T. Corcoran (Eds.), *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction* (Research Report No. RR-68; pp. 35–40). Consortium for Policy Research in Education. Retrieved from www.cpre.org/images/stories/cpre_pdfs/learning%20trajectories%20in%20math_ccii%20report.pdf.
	Percents	Middle school	Pöhler, B., & Prediger, S. (2015). Intertwining lexical and conceptual learning trajectories: A design research study on dual macro-scaffolding towards percentages. *Eurasia Journal of Mathematics, Science & Technology Education*, 11(6), 1697–1722.
	Whole Numbers and Operations	Elementary school	Roy, G. J. (2008). Prospective teachers' development of whole number concepts and operations during a classroom teaching experiment (Doctoral dissertation). Retrieved from http://etd.fcla.edu/CF/CFE0002398/Roy_George_J_200812_PhD.pdf
	Fractions	Elementary school	Saenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education*, 50-85.
	Representing Integers on a Number Line	Elementary school	Saxe, G. B., Earnest, D., Sitabkhan, Y., Haldar, L. C., Lewis, K. E., & Zheng, Y. (2010). Supporting generative thinking about integers on number lines in elementary mathematics. *Cognition and Instruction*, 28(4), 433–474.
	Representing Fractions on a Number Line	Elementary and Middle school	Saxe, G. B., Shaughnessy, M. M., Shannon, A., Langer-Osuna, J. M., Chinn, R., & Gearhart, M. (2007). Learning about fractions as points on a number line. In W. G. Martin, M. E. Strutchens, & P. C. Elliott, (Eds.), *The Learning of Mathematics: 2007 Yearbook* (pp. 221–237). Reston, VA: NCTM.
	Fractional Notation and Representation	Elementary school	Saxe, G. B., Taylor, E. V., McIntosh, C., & Gearhart, M. (2005). Representing fractions with standard notation: A developmental analysis. *Journal for Research in Mathematics Education*, 137-157.
	Addition and Multiplication	Elementary school	Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for Research in Mathematics Education,* 36(4), 347–395. doi:10.2307/30035044
	Multiplicative Reasoning	Elementary and Middle school	Siemon, D. (2018). Knowing and Building on What Students Know – The Case of Multiplicative Thinking. In D. Siemon, T. Barkatsas & R. Seah (Eds.), *Researching and using learning progressions (trajectories) in mathematics education.* Leiden, the Netherlands: SENSE Publishers.
	Fractions	Elementary school	Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical thinking and learning*, 6(2), 91-104.
	Decimals	Elementary, Middle, and High school	Stacey, K. and Steinle, V. (1999). A Longitudinal Study of Children's Thinking about decimals: A preliminary Analysis. In O. Zaslavsky (Ed.), *Proceedings from 23rd Conference of the International Group for Psychology of Mathematics Education.* Vol4. (pp 233-240) Haifa, Israel: PME

	Decimal Notation	Elementary, Middle, and High school	Stacey, K., & Steinle, V. (2006). A case of the inapplicability of the Rasch model: Mapping conceptual learning. *Mathematics Education Research Journal*, 18(2), 77–92. doi:10.1007 /BF03217437
	Proportional Reasoning	Elementary school	Steinthorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5th grade girls' developmental trajectories in proportional reasoning. *Mathematics Education Research Journal*, 21(1), 6–30. doi:10.1007/BF03217536
	Integer Addition and Subtraction	Middle school	Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education*, 43(4), 428–464. doi:10.5951 /jresematheduc.43.4.0428
	Number operations	Elementary school	Stephens, M., & Armanto, D. (2010). How to build powerful learning trajectories for relational thinking in the primary school years. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), Shaping the future of mathematics education: *Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia*. (pp. 523–530). Fremantle, Australia: MERGA. Retrieved from http://files.eric.ed.gov /fulltext/ED520968.pdf
	Ratio	Elementary school	Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process (towards... a theory). *Educational Studies in Mathematics*, 15(4), 327-348.
	Negative Numbers	Elementary school	Streefland, L. (1996). Negative numbers: Reflections of a learning researcher. *The Journal of Mathematical Behavior*, 15(1), 57-77.
	Percents	Elementary and Middle school	Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics* 54(1), 9-35. Van den Heuvel-Panhuizen, M., Middleton, J. A., & Streefland, L. (1995). Student-generated problems: Easy and difficult problems on percentage. *For the Learning of Mathematics*, 21-27.
	Fractions, decimals, percentages, proportions	Elementary school	Van Galen, F., Feijs, E., Figueiredo, N., Gravemeijer, K., van Herpen, E., & Keijzer, R. (2008). *Fractions, Percentages, Decimals and Proportions: A Learning-Teaching Trajectory for Grade 4, 5 and 6*. Sense Publishers, Rotterdam.
	Multiplicative Reasoning	Middle school	Venkat, H., & Mathews, C. (2018). Improving multiplicative reasoning in a context of low performance. *ZDM*, 1-14.
	Proportional Reasoning	Middle school	*Vermont Mathematics Partnership's Ongoing Assessment Project*. (2013). OGAP proportional reasoning framework. Montpelier, VT: Author. Retrieved from http://margepetit.com/wp-content/uploads/2015/04/OGAPProportionalFramework10.2013.pdf
	Fractions	Middle school	Vermont Mathematics Partnership's Ongoing Assessment Project. (2014a). OGAP fraction framework. Montpelier, VT: Author. Retrieved from www.ogapmath.com/wp-content/uploads/2017/04/Fraction-Framework-Color-11x17-01.16.14.pdf

	Addition	Middle school	*Vermont Mathematics Partnership's Ongoing Assessment Project*. (2014b). OGAP additive framework. Montpelier, VT: Author. Retrieved from www.ogapmath.com/wp-content/uploads/2017/04/framework_November2017.pdf
	Multiplication	Middle school	*Vermont Mathematics Partnership's Ongoing Assessment Project.* (2014b). OGAP multiplicative framework. Montpelier, VT: Author. Retrieved from www.ogapmath.com/wp-content/uploads/2017/04/OGAP-Multiplicative-Framework-Color-1.11.2017.pdf Ebby, C., Sirinides, P., & Supovitz, J. (2017). Developing measures of teacher and student understanding in relation to learning trajectories. Paper presented at the *2017 Annual Meeting of the American Educational Research Association;* San Antonio, TX.
	Rational number reasoning	Middle school	Wright, V. (2014). Towards a hypothetical learning trajectory for rational number. *Mathematics Education Research Journal* 26(3), 635-657.

STATISTICS AND PROBABILITY	Reasoning about variability	Middle school	Ben-Zvi, D. (2004). Reasoning about variability in comparing distributions. *Statistics Education Research Journal* 3(2) 42-63.
	Describing distributions	Elementary school	Leavy, A. M., & Middleton, J. A. (2011). Elementary and middle grade students' constructions of typicality. *The Journal of Mathematical Behavior*, 30(3), 235-254.
	Statistical reasoning	Middle school	Lehrer, R., Kim, M. J., Ayers, E., & Wilson, M. (2014). Toward establishing a learning progression to support the development of statistical reasoning. *Learning over time: Learning trajectories in mathematics education*. Charlotte, NC: Information Age. Shinohara, M., & Lehrer, R. (2018). Becoming Statistical. Annual Meeting of the American Education Research Association. New York, NY. April 13..
	Inference	Elementary school	Makar, K. & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal*, 8(1), 82-105.
	Early statistical reasoning	Middle school	McGatha, M., Cobb, P., & McClain, K. (2002). An analysis of students' initial statistical understandings: Developing a conjectured learning trajectory. *The Journal of Mathematical Behavior*, 21(3), 339-355.
	Sampling	Middle school	Meletiou-Mavrotheris, M., & Paparistodemou, E. (2015). Developing students' reasoning about samples and sampling in the context of informal inferences. *Educational Studies in Mathematics*, 88(3), 385–404.
	Probability	Middle school	Watson, J. M., & Kelly, B. A. (2009). Development of student understanding of outcomes involving two or more dice. *International Journal of Science and Mathematics Education*, 7(1), 25-54.
	Inference	Middle school	Zieffler, A., Garfield, J., Delmas, R., & Reading, C. (2008). A framework to support research on informal inferential reasoning. *Statistics Education Research Journal* 7(2), 40-58.

Appendix B. Theoretical Publications and Studies of Applications of LT/LPs in Mathematics

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. Mathematical Thinking and Learning, 6(2), 81–89. doi:10.1207/s15327833mt10602_1

Clements, D. H., & Sarama, J. (2009). Learning and teaching early math: The learning trajectories approach. New York, NY: Routledge.

Clements, D. H., & Sarama, J. (2014). Learning trajectories: Foundations for effective, research-based education. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), Learning over time: Learning trajectories in mathematics education(pp. 1–30). Charlotte, NC: Information Age.

Common Core Standards Writing Team. (2013a). A progression of the Common Core State Standards in Mathematics (draft). Tucson, AZ: Institute for Mathematics and Education, University of Arizona. Retrieved from http://commoncoretools.me/wp-content/uploads/2011/08/ccss_progression_nf_35_2013_09_19.pdf

Confrey, J., Gianopulos, G., McGowan, W., Shah, M., & Belcher, M. (2017). Scaffolding learner-centered curricular coherence using learning maps and diagnostic assessments designed around mathematics learning trajectories. ZDM, , 1-18. 10.1007/s11858-017-0869-1

Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: A framework for connecting standards with curriculum. ZDM—The International Journal on Mathematics Education, 46(5), 719–733. doi:10.1007/s11858-014-0598-7

Confrey, J., Maloney, A. P., & Nguyen, K. H. (2014). Learning trajectories in mathematics. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), Learning over time: Learning trajectories in mathematics education (pp. xi–xxi). Charlotte, NC: Information Age.

Delgado, C., & Morton, K.* (2012). Learning progressions, learning trajectories, and equity. In van Aalst, J., Thompson, K., Jacobson, M. J., & Reimann, P. (Eds.) (2012). The future of learning: Proceedings of the 10th International Conference of the Learning Sciences (ICLS 2012) – Volume 1, Full papers, pp. 204-211. International Society of the Learning Sciences: Sydney, NSW, Australia.

Empson, S. (2011). On the idea of learning trajectories: Promises and pitfalls. The Mathematics Enthusiast, 8(3), 571–596.

Heritage, M. (2008). Learning progressions: Supporting instruction and formative assessment. Washington, DC: The Council of Chief State School Officers.

Heritage, M. (2009). The case for learning progressions. San Francisco, CA: Stupski Foundation

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Leahy, S., & Wiliam, D. (2011). Devising learning progressions assessment. In Annual Meeting of the American Educational Research Association, New Orleans, LA.

Lehrer, R., Jones, R. S., Pfaff, E., & Shinohara, M. (2017). Data Modeling Supports the Development of Statistical Reasoning. Final report submitted to the Institute of Education Sciences, U.S. Department of Education. Washington, DC

Maloney, A. P., Confrey, J., & Nguyen, K. H. (Eds.). (2014). Learning over time: Learning trajectories in mathematics education. IAP.

Sarama, J., & Clements, D. H. (2009). Early childhood mathematics education research: Learning trajectories for young children. New York: NY: Routledge.

Sarama, J., & Clements, D. H. (2013). Lessons learned in the implementation of the TRIAD scale-up model: Teaching early mathematics with trajectories and technologies. In T. Halle, A. J. R. Metz & I. Martinez-Beck (Eds.), Applying implementation science in early childhood programs and systems (pp. 173-191). Baltimore, MD: Paul H. Brookes.

Sarama, J., Clements, D. H., Starkey, P., Klein, A., & Wakeley, A. (2008). Scaling up the implementation of a pre-kindergarten mathematics curriculum: Teaching for understanding with trajectories and technologies. Journal of Research on Educational Effectiveness, 1(2), 89-119. 10.1080/19345740801941332 Retrieved from http://dx.doi.org/10.1080/19345740801941332

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Siemon, D. (2017). Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years: Introducing the Reframing Mathematical Futures II Project. In The 40th Annual Conference of the Mathematics Education Research Group of Australasia (pp. 651-654). The Mathematics Education Research Group of Australasia Inc.

Simon, M. A., Saldanha, L., McClintock, E., Akar, G. K., Watanabe, T., & Zembat, I. O. (2010). A developing approach to studying students' learning through their mathematical activity. Cognition and Instruction, 28(1), 70–112. doi:10.1080/07370000903430566

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Wiser, M., Smith, C. L., & Doubler, S. (2012). Learning progressions as tools for curriculum development. In A. C. Alonzo & A. W. Gotwals (Eds.), Learning progressions in science: Current challenges and future directions (pp. 359–403). Rotterdam, The Netherlands: Sense.

Yettick, H. (2015). Learning progressions: Maps to personalized teaching. Education Week, 35(12), S18–S19. Retrieved from www.edweek.org/ew/articles/2015/11/11/learning-progressions-maps-to-personalized-teaching.html

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